JEE Adv. May 2026

Question Paper With Text Solution

17 May | Paper-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911
Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE ADV. MAY 2026 | 26$^{TH.}$ MAY PAPER-2

SECTION – 1 (MAXIMUM MARKS: 12)

- This section contains **FOUR (04)** question stems.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –1 In all other cases.

1. Let $\vec{a}, \vec{b}$ be two vectors, and let P, Q and R be the points with position vectors $\vec{a}, \vec{b}$ and $\vec{a}+\vec{b}$, respectively, with respect to the origin O. If $|\vec{a}+\vec{b}| = \sqrt{21}, |\vec{a}-\vec{b}| = 3$, and $\vec{a}$ and $(\vec{a}-\vec{b})$ are perpendicular to each other, then the area of the triangle OPR is :

(A) $\sqrt{3}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{3\sqrt{3}}{2}$ (D) $\frac{3}{2}$

Ans. C

Sol. $P(\vec{a}), Q(\vec{b}), R(\vec{a}+\vec{b})$

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2.\vec{a}.\vec{b} = 21$$

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2.\vec{a}.\vec{b} = 9$$

$$|\vec{a}|^2 + |\vec{b}|^2 = 15$$

$$\vec{a}.\vec{b} = 3$$

also, $\vec{a}.(\vec{a}-\vec{b}) = 0$

$$|\vec{a}|^2 = \vec{a}.\vec{b} = 3$$

$$|\vec{b}|^2 = 12$$

$$\Delta_{OPR} = \frac{1}{2}\left|\overrightarrow{OP} \times \overrightarrow{OR}\right|$$

$$\Delta_{OPR} = \frac{1}{2}\left|\vec{a} \times \left(\vec{a} + \vec{b}\right)\right| \quad = \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{1}{2}\sqrt{(\vec{a})^2 (\vec{b})^2 - (\vec{a}.\vec{b})^2} \quad = \frac{1}{2}\sqrt{36 - 9} = \frac{3\sqrt{3}}{2}$$

2. Let T be the tangent to the parabola $y^2 = 16x$ at the point (64, 32). Let L be the tangent to the same parabola at another point (x_1, y_1) on the parabola. If L and T are perpendicular to each other, then the distance between the point (x_1, y_1) and the focus of the parabola, is :

(A) $\frac{15}{4}$ (B) 4 (C) $\frac{17}{4}$ (D) 5

Ans. C

Sol. P : $y^2 = 16x$

Tangent at (64, 32) $\Rightarrow$ T = 0

$y.32 = 8(x + 64) \Rightarrow \quad 4y = x + 64$

$m_T = \frac{1}{4} \Rightarrow \quad m_L = -4$

$(x_1, y_1) = (at^2, 2at) = \left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{16}, \frac{8}{-4}\right) = \left(\frac{1}{4}, -2\right)$

Focal distance $= x + a = \frac{1}{4} + 4 = \frac{17}{4}$

3. Let $y : (-\infty, \infty) \to (0, \infty)$ be the solution of the differential equation $\frac{dy}{dx} = \frac{e^{5x}y^3 + y^3}{e^x + e^x y^4}$, satisfying $y(0) = \frac{1}{\sqrt{2}}$.

Then the value of $y(\log_e 2)$ is :

क :

(A) $\sqrt{\frac{5+\sqrt{35}}{2}}$ (B) $\sqrt{\frac{7+\sqrt{53}}{2}}$ (C) $\frac{7+\sqrt{53}}{2}$ (D) $\frac{5+\sqrt{35}}{2}$

Ans. B

Sol. $\frac{dy}{dx}=\frac{\left(e^{5x}+1\right)y^3}{e^x\left(1+y^4\right)}$

$\int\frac{1+y^4}{y^3}dy=\int\frac{e^{5x}+1}{e^x}dx$

$\Rightarrow\frac{-1}{2y^2}+\frac{y^2}{2}=\frac{e^{4x}}{4}-e^{-x}+C$

$y(0)=\frac{1}{\sqrt{2}}\Rightarrow-1+\frac{1}{4}=\frac{1}{4}-1+C\Rightarrow C=0$

$-\frac{1}{2y^2}+\frac{y^2}{2}=\frac{e^{4x}}{4}-\frac{1}{e^x}$

$x=\log_e^2\Rightarrow\frac{-1}{2y^2}+\frac{y^2}{2}=\frac{e^{4\log_e 2}}{4}-\frac{1}{e^{\log_e 2}}$

$-\frac{1}{2y^2}+\frac{y^2}{2}=4-\frac{1}{2}=\frac{7}{2}$

$-1+y^4=7y^2$

$y^x-7y^2-1=0$

$y^2=\frac{7\pm\sqrt{49+4}}{2}$

$y^2=\frac{7+\sqrt{53}}{2}$

$y=\sqrt{\frac{7+\sqrt{53}}{2}}$

4. The value of the definite integral $\int_0^2\frac{1}{3^x+3}dx$ is :

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{\log_e 3}{3}$ (D) $\frac{\log_e 3}{2}$

Ans. B

Sol. $I=\int_0^2 \frac{1}{3x+3}dx$

$3^x = t \Rightarrow 3^x \log_e 3\ dx = dt$

$dx = \frac{1}{\log_e 3 \cdot t}dt$

$I=\int_1^9 \frac{1}{t+3}\cdot\frac{dt}{t.\log_e 3} \Rightarrow \quad I=\frac{1}{\log_e 3}\int_1^9\left(\frac{-\frac{1}{3}}{t+3}+\frac{\frac{1}{3}}{t}\right)dt$

$I=\frac{1}{\log_e 3}\cdot\frac{1}{3}\log_e\left|\frac{t}{t+3}\right|_1^9 \Rightarrow \quad I=\frac{1}{3\log_e 3}\left\{\log_e\left(\frac{9}{12}\right)-\log_e\left(\frac{1}{4}\right)\right\}$

$I=\frac{1}{3\log_e 3}\cdot \ell n_e 3=\frac{1}{3}$

SECTION – 2 (MAXIMUM MARKS: 20)

- This section contains **FIVE (05)** question stems.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks	: +4 **ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks	: +3 If all the four options are correct but ONLY three options are chosen;
Partial Marks	: +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks	: +1 If two or more options are correct but ONLY one option is chosen and it is acorrect option;
Zero Marks	: 0 If unanswered;
Negative Marks	: –1 In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

  choosing ONLY (B) will get +1 mark;

  choosing ONLY (D) will get +1 mark;

  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get -1 marks.

5. Let R denote the set of all real numbers. Consider the polynomial function $f : R \to R$ defined by

$$f(x) = \frac{d^{10}}{dx^{10}}((x^2-1)^{10}), \quad \text{for all } x \in \mathbb{R}.$$

Here $\frac{d^{10}}{dx^{10}}((x^2-1)^{10})$ is the 10^{th} order derivative of the function $(x^2 - 1)^{10}$.

Then which of the following statements is (are) **TRUE** :

(A) The coefficient of x^8 in the polynomial f(x) is $(-10)\left(\frac{18!}{8!}\right)$

(B) The value of $f(1) + f(-1)$ is equal to $10!2^{11}$

(C) The degree of the polynomial f(x) is 10

(D) The constant term of the polynomial f(x) is $-\left(\frac{10!}{5!}\right)$

Ans. ABC

Sol. Let $g(x) = (x^2 - 1)^{10}$

$g(x) = x^{20} - {}^{10}C_1\, x^{18} + {}^{10}C_2\, x^{16} + \ldots. - {}^{10}C_5\, x^{10} + {}^{10}C_6\, x^8 \ldots.. + 1$

$$f(x) = \frac{20!}{10!}x^{10} - {}^{10}C_1 \frac{18!}{8!}x^8 + {}^{10}C_2 \frac{16!}{8!}x^6 \ldots\ldots {}^{10}C_5 \cdot 10!$$

Coefficient of $x^8 = -10\cdot\frac{18!}{8!}$

Degree = 10

Constant term = $-{}^{10}C_5\cdot 10! = -\left(\frac{10!}{5!}\right)^2$

$g(x) = (x^2 - 1)^{10}$

$g'(x) = 10(x^2 - 1)^9\cdot 2x \Rightarrow g'(1) = 0$

$g''(x) = 2\cdot 10(x^2 - 1)^9 + 2^2\cdot 10\cdot 9(x^2 - 1)^8\cdot x \Rightarrow g''(1)$

For f(1) or f(–1) all terms becomes 0 except the last term.

$f(1) = f(-1) = 2^{10}\cdot 10!$

$f(1) + f(-1) = 2^{11}\cdot 10!$

6. Let a, b, c be positive integers in arithmetic progression such that the equation

$$ax^2 + bx + c = 0$$

has only integer solutions.

Then which of the following statements is (are) **TRUE** :

(A) c – b is an integer multiple of a

(B) Both the roots of the equation $ax^2 + bx + c = 0$ are odd integers

(C) If c = 15, then ab = 8

(D) If b = 8, then x = 3 is a root of the equation $ax^2 + bx + c = 0$

Ans. ABC

Sol. a, b, c are in A.P. So,

$2b = a + c$

Now, $ax^2 + bx + c = 0 \genfrac{\langle}{}{0pt}{}{\alpha}{\beta}$

$ax^2 + bx + c = a(x - \alpha)(x - \beta)$

put x = –2

$4a - 2b + c = a(-2 - \alpha)(-2 - \beta)$

$4a - (a + c) + c = a(\alpha + 2)(\beta + 2)$

$(\alpha + 2)(\beta + 2) = 3$

Because a, b, c > 0 so $\alpha, \beta < 0$ and α, β are integers.

$(\alpha + 2)(\beta + 2) = (-3)(-1)$

$(\alpha, \beta) \equiv (-3, -5)$ OR $(-5, -3)$

so, b = 8a (S.O.R.)

c = 15a (P.O.R.)

7. Let L be the straight line joining the points P(1, 2, –1) and Q(2, 3, 1). Let S be the foot of the perpendicular drawn from the point R(4, –1, 5) to the line L. Another line passing through R intersects L at a point T such that the point S divides the line segment PT internally in the ratio |PS| : |ST| = 1 : 2, where |PS| and |ST| are the lengths of the line segments PS and ST, respectively.

Then which of the following statements is (are) **TRUE** :

(A) The orthocentre of the triangle PRT is $\left(\frac{23}{5}, -4, \frac{31}{5}\right)$

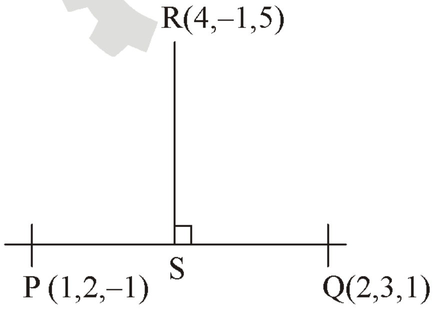
(B) The orthocentre of the triangle PRT is (4, 3, 5)

(C) The area of the triangle PRT is $6\sqrt{5}$

(D) The area of the triangle PRT is $18\sqrt{5}$

Ans. AD

Sol.



Equation of Line L : $\overline{r} = (1, 2, -1) + \lambda(1, 1, 2)$

Let point $S \equiv (1 + \lambda, 2 + \lambda, -1 + 2\lambda)$

For "λ"

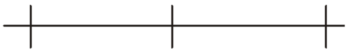
$\overrightarrow{RS}\cdot\left(\hat{i}+\hat{j}+2\hat{k}\right)=0$

$1(\lambda-3)+1\,(\lambda+3)+2\,(2\lambda-6)=0$

$\lambda=2$

S (3, 4, 3)

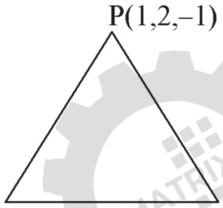
Now, $|PS| = |ST| = 1 : 2$



$\dfrac{2+x}{3}=3\Rightarrow x=7$

$\dfrac{4+y}{3}=4\Rightarrow y=8$ T(7,8,11)

$\dfrac{-2+z}{3}=3\Rightarrow z=11$



Area of $\Delta PRT=\dfrac{1}{2}\left|\overline{PR}\times\overline{RT}\right|$

$\overline{PR}=3\hat{i}-3\hat{j}+6\hat{k}$

$\overline{RT}=3\hat{i}+9\hat{j}+6\hat{k}$

$$\overline{PR}\times\overline{RT}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -3 & 6\\ 3 & 9 & 6\end{vmatrix}$$

$=-72\hat{i}+36\hat{k}=36\left(-2\hat{i}+\hat{k}\right)$

$\Delta = \frac{1}{2} \times 36\sqrt{5} = 18\sqrt{5}$

for orthocentre $\overline{RS} \perp PT$, So

H lies on line RS

& $\overline{RS} \perp \overline{RT}$

H$(4 - \mu, -1 + 5\mu, 5 - 2\mu)$

$\overrightarrow{PH}.\overline{RT} = 0$

$\mu = -\frac{3}{5}$

$H\left(\frac{23}{5}, -4, \frac{31}{5}\right)$

8. Let y = f(x) be the real valued function defined on the interval $(0, \infty)$, satisfying $y(1) = 0$ and the differential equation

$$x\frac{dy}{dx} = y - x^3$$

Then which of the following statements is (are) **TRUE** :

(A) The function f has a local minimum at $x = \frac{1}{\sqrt{3}}$

(B) The function f has a local maximum at $x = \frac{1}{\sqrt{3}}$

(C) The function f is increasing in the interval (1, 2)

(D) If $g(x) = 4x^3 - 5x^2 + \frac{3}{2}x$ for $x > 0$, then the number of elements in the set $\{x \in (0, \infty) : f(x) = g(x)\}$ is 2

Ans. BD

Sol. $\frac{xdy}{dx} = y - x^3$

$\frac{dy}{dx} - \frac{1}{x}y = -x^2$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} \Rightarrow e^{-\ell n(x)} \Rightarrow \frac{1}{x}$$

Solution $\quad y \times \frac{1}{x} = \int \frac{1}{x}\left(-x^2\right)dx + C$

$$\frac{y}{x} = -\frac{x^2}{2} + C$$

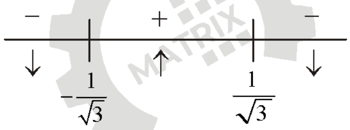
$$y = -\frac{x^3}{2} + Cx$$

$$y(1) = 0 \Rightarrow c = \frac{1}{2}$$

$$y = -\frac{x^3}{2} + \frac{x}{2}$$

$$\frac{dy}{dx} = -\frac{3x^2}{2} + \frac{1}{2}$$

$$\Rightarrow -\frac{3}{2}\left(x - \frac{1}{\sqrt{3}}\right)\left(x + \frac{1}{\sqrt{3}}\right)$$



Local maxima at $x = \frac{1}{\sqrt{3}}$

Local minima at $x = -\frac{1}{\sqrt{3}}$

For f(x) = g(x)

$$-\frac{x^3}{2} + \frac{x}{2} = 4x^3 - 5x^2 + \frac{3}{2}x$$

$$9x^3 - 10x^2 + 2x = 0$$

$x(9x^2 - 10x + 2) = 0$

$x = 0,\ x = \dfrac{5 \pm \sqrt{7}}{2}$

For $x > 0$, only 2 values of x.

9. Let R denote the set of all real numbers and let $i = \sqrt{-1}$. Consider the matrices

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Let a, b, c, d be real numbers such that

$$ST = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let

$$H = \{x + iy : x, y \in R \text{ and } y > 0\}.$$

Then which of the following statements is (are) TRUE :

(A) $\dfrac{b+ia}{d+ic} = i$

(B) If $\omega = \dfrac{-1+i\sqrt{3}}{2}$, then $\dfrac{a\omega+b}{c\omega+d} = \omega$

(C) If m is an integer greater than 2 such that $(ST)^2 = (ST)^m$, then m is an integer multiple of 8

(D) If $z \in H$, then $\dfrac{az+b}{cz+d} \in H$

Ans. BD

Sol. $ST = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a = 0,\ b = -1,\ c = 1,\ d = 1$

(A) $\dfrac{b+ia}{d+ic} = \dfrac{-1}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{i-1}{2}$ (Incorrect)

(B) $\frac{a\omega+b}{c\omega+d}=\frac{-1}{\omega+1}=\frac{-1}{-\omega^2}=\omega\,(\text{correct})$

(C) $(ST)^2=\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}$

$(ST)^3=\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=-I$

$(ST)^6 = I$

$(ST)^2 = (ST)^M$

where $M = 6\lambda + 2$ (Incorrect)

(D) $\frac{az+b}{cz+d}=\frac{-1}{z+1}=\frac{-1}{(x+1)+iy}\times\frac{(x+1)-iy}{(x+1)-iy}$

$=-\frac{(x+1)+iy}{(x+1)^2+y^2}$

Imaginary part $=\frac{y}{(x+1)^2+y^2}>0$ as $y>0$

SECTION – 3 (MAXIMUM MARKS: 20)

- This section contains **FIVE (05)** question stems.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;

Zero Marks : 0 In all other cases.

10. Let N denote the set of all positive integers. Consider the sets

$A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\}$

Let S be the set of all functions $f : A \rightarrow B$ such that $f(2) \neq 2$ and $f(4) \neq 4$. Consider the set

$$T = \{f \in S : \text{there exists a function } g : B \rightarrow N \text{ such that } g(f(x)) = 2^x \text{ for all } x \in A\}$$

Then the number of elements in the set T is _______.

Ans. 1860

Sol. $g(f(x)) = 2^x \ \forall \ x \in A$

because 2^x is one-one function.

So, f(x) must be one-one function.

So, total no. of one-one function $= 7 \times 6 \times 5 \times 4 \times 3 = 2520$

but $f(2) \neq 2$ and $f(4) \neq 4$

So, required function = Total – (when f(2) = 2 OR f(4) = 4) + (when f(2) = 2 and f(4) = 4)

$= 2520 - ({}^6C_4 \times 4! + {}^6C_4 \times 4!) + {}^5C_3 \times 3!$

$= 2520 - (360 + 360) + 60$

$= 1860$

11. A bookshelf contains 6 distinct books of Mathematics and 5 distinct books of Physics. From these 11 books, 6 books are chosen at random. Let X be the absolute value of the difference between the number of Mathematics books chosen and the number of Physics books chosen. If α is the mean of the random variable X, then the value of 77α is _______.

Ans. 100

Sol. Total number of Mathematics books = 6

Total number of Physics books = 5

Total books = 11

Number of books chosen = 6

$$\text{Total Ways} = \binom{11}{6} = \frac{11 \times 10 \times 9 \times 8 \times 7}{5 \times 4 \times 3 \times 2 \times 1} = 462$$

Let M be the number of Mathematics books chosen, and P be the number of Physics books chosen.

Since we select 6 books in total $\quad M + P = 6 \Rightarrow M = 6 - P$

The random variable X is defined as the absolute difference: $X = |M - P| = |(6 - P) - P| = |6 - 2P|$

| Physics books (P) | Mathematics books (M) | Number of Ways: $\binom{6}{M} \times \binom{5}{P}$ | Value of $X = |M - P|$ |

P = 0; Number of ways = 1 ; X = 6

P = 1: Number of ways = 30 ; X = 4

P = 2: Number of ways = 150, X = 2

P = 3: Number of ways = 200; X = 0

P = 4: Number of ways = 75; X = 2

P = 5: Number of ways = 6 ; X = 4

Summing up the ways for identical values of X:

For X = 6: Total ways = 1

For X = 4: Total ways = 30 + 6 = 36

For X = 2: Total ways = 150 + 75 = 225

For X = 0: Total ways = 200

$$\alpha = \sum X \cdot P(X)$$

$$\alpha = \left(0 \times \frac{200}{462}\right) + \left(2 \times \frac{225}{462}\right) + \left(4 \times \frac{36}{462}\right) + \left(6 \times \frac{1}{462}\right)$$

$$\alpha = \frac{0 + 450 + 144 + 6}{462} \Rightarrow \alpha = \frac{100}{77}$$

12. Consider a data consisting of 10 observations $x_1, x_2, \ldots\ldots\ldots, x_{10}$, whose mean is 5 and variance is 7. If the mean and the variance of the first 8 observations $x_1, x_2, \ldots, x_8$ are 4 and 3.5, respectively, and $x_9 < x_{10}$, then the value of $3x_9 + 2x_{10}$ is ________.

Ans. 44

Sol. The mean of n observations is given by $\overline{x} = \frac{\sum x_i}{n}$

$$\overline{x}_{10} = \frac{\sum_{i=1}^{10} x_i}{10} = 5 \Rightarrow \sum_{i=1}^{10} x_i = 50$$

For the first 8 observations: $\overline{x}_8 = \frac{\sum_{i=1}^{8} x_i}{8} = 4 \Rightarrow \sum_{i=1}^{8} x_i = 32$

$\sum_{i=1}^{10} x_i = \sum_{i=1}^{8} x_i + x_9 + x_{10} \Rightarrow \qquad 50 = 32 + x_9 + x_{10}$

$x_9 + x_{10} = 18$ --- (Equation 1)

$\sigma^2 = \frac{\sum x_i^2}{n} - (\overline{x})^2$

For the first 8 observations: $\sigma_8^2 = \frac{\sum_{i=1}^{8} x_i^2}{8} - (\overline{x}_8)^2$

$3.5 = \frac{\sum_{i=1}^{8} x_i^2}{8} - 4^2 \Rightarrow \quad 3.5 = \frac{\sum_{i=1}^{8} x_i^2}{8} - 16 \Rightarrow \quad 19.5 = \frac{\sum_{i=1}^{8} x_i^2}{8} \Rightarrow \sum_{i=1}^{8} x_i^2 = 19.5 \times 8 = 156$

For all 10 observations: $\sigma_{10}^2 = \frac{\sum_{i=1}^{10} x_i^2}{10} - (\overline{x}_{10})^2$

$7 = \frac{\sum_{i=1}^{10} x_i^2}{10} - 5^2 \Rightarrow 7 = \frac{\sum_{i=1}^{10} x_i^2}{10} - 25 \Rightarrow \quad 32 = \frac{\sum_{i=1}^{10} x_i^2}{10} \Rightarrow \sum_{i=1}^{10} x_i^2 = 320$

$\sum_{i=1}^{10} x_i^2 = \sum_{i=1}^{8} x_i^2 + x_9^2 + x_{10}^2$

$x_9^2 + x_{10}^2 = 320 - 156$

$x_9^2 + x_{10}^2 = 164$ --- (Equation 2)

$(x_9 + x_{10})^2 = x_9^2 + x_{10}^2 + 2x_9x_{10}$:

$18^2 = 164 + 2x_9x_{10} \Rightarrow \quad 324 = 164 + 2x_9x_{10} \Rightarrow \quad 2x_9x_{10} = 324 - 164 = 160$

$\Rightarrow x_{10} = 10$

Substituting $x_{10} = 10$ back into Equation 1: $x_9 + 10 = 18 \Rightarrow x_9 = 8$

Value $= 3x_9 + 2x_{10} = 44$

13. Consider the ellipse E given by $\frac{x^2}{18}+\frac{y^2}{12}=1$. Let H be the hyperbola whose eccentricity is the reciprocal of the eccentricity of E and whose foci are the same as that of E. Let P and Q be the points of intersection of H and the parabola $\sqrt{5}y = x^2$ in the first quadrant. Let d be the distance between P and Q.

If a and b are the integers such that $d^2 = a + b\sqrt{5}$, then the value of $a - b$ is _______.

Ans. 18

Sol. The given equation of the ellipse is: $E : \frac{x^2}{18}+\frac{y^2}{12}=1$

$a_e^2 = 18$ and $b_e^2 = 12$

The eccentricity e_E of the ellipse is $e_E = \sqrt{1-\frac{b_e^2}{a_e^2}} = \sqrt{1-\frac{12}{18}} = \sqrt{1-\frac{2}{3}} = \frac{1}{\sqrt{3}}$

$a_e e_E = \sqrt{18}\cdot\frac{1}{\sqrt{3}} = \sqrt{6}$; Thus, the foci are $(\pm\sqrt{6}, 0)$

Let the equation of the hyperbola be $\frac{x^2}{a_h^2}-\frac{y^2}{b_h^2}=1$

$e_H = \frac{1}{e_E} = \sqrt{3}$

Foci of H: The foci are identical to those of the ellipse, so $a_h e_H = \sqrt{6}$

$a_h \cdot \sqrt{3} = \sqrt{6} \Rightarrow a_h = \sqrt{2} \Rightarrow a_h^2 = 2$

$b_h^2 = a_h^2(e_H^2 - 1) = 2((\sqrt{3})^2 - 1) = 2(3-1) = 4$

$H : \frac{x^2}{2}-\frac{y^2}{4}=1$

The given equation of the parabola is: $\sqrt{5}y = x^2 \Rightarrow x^2 = \sqrt{5}y$

Substitute $x^2 = \sqrt{5}y$ into the equation of the hyperbola H

$$\frac{\sqrt{5}y}{2} - \frac{y^2}{4} = 1$$

$$2\sqrt{5}y - y^2 = 4 \Rightarrow y^2 - 2\sqrt{5}y + 4 = 0$$

$$y = \frac{2\sqrt{5} \pm \sqrt{(-2\sqrt{5})^2 - 4(1)(4)}}{2} = \frac{2\sqrt{5} \pm \sqrt{20-16}}{2} = \frac{2\sqrt{5} \pm 2}{2} = \sqrt{5} \pm 1$$

Since both values are positive, we have two points in the first quadrant

$$y_1 = \sqrt{5} + 1 \Rightarrow x_1^2 = \sqrt{5}(\sqrt{5}+1) = 5 + \sqrt{5}$$

$$y_2 = \sqrt{5} - 1 \Rightarrow x_2^2 = \sqrt{5}(\sqrt{5}-1) = 5 - \sqrt{5}$$

Thus, the intersection points P and Q in the first quadrant $(x > 0)$ are

$$P = \left(\sqrt{5+\sqrt{5}}, \sqrt{5}+1\right) \quad \text{and} \quad Q = \left(\sqrt{5-\sqrt{5}}, \sqrt{5}-1\right)$$

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$d^2 = (10 - 4\sqrt{5}) + 4 = 14 - 4\sqrt{5}$$

$$a = 14 \quad \text{and} \quad b = -4$$

$$a - b = 14 - (-4) = 14 + 4 = 18$$

14. For a real number α, let $[\alpha]$ denote the greatest integer less than or equal to α. For a finite set S, let $|S|$ denote the number of elements in the set S.

Consider the functions $f : (-3,3) \to (-\infty, \infty)$ and $g : (-3,3) \to (-\infty, \infty)$ defined by

$$f(x) = [x^3]\log_e\left(1 + \sin^2(\pi(x - [x]))\right) \text{ and}$$

$$g(x) = x^3 \sin^2\left(\pi \log_e(1 + x - [x])\right).$$

Let $A = \{x \in (-3, 3) : f \text{ is discontinuous at } x\}$ and $B = \{x \in (-3, 3) : g \text{ is discontinuous at } x\}$

Then the value of $|A| + 2|B| - |A \cap B|$ is ________.

Ans. 56

Sol. The function is given by: $f(x) = [x^3]\log_e\left(1+\sin^2(\pi\{x\})\right)$

The potential points of discontinuity for f(x) in the interval (–3, 3) are where either {x} or $[x^3]$ is discontinuous

{x} is discontinuous at all integers $x \in Z$

$[x^3]$ is discontinuous where $x^3 \in Z$

Case I: At integer points $(x = k \in Z)$

For $x \in (-3, 3)$, the integer points are $k \in \{-2, -1, 0, 1, 2\}$. Let's check the continuity at x = k

$$f(k) = [k^3]\log_e(1+\sin^2(\pi\cdot 0)) = [k^3]\log_e(1) = 0$$

$$\lim_{x\to k^+} f(x) = k^3 \log_e(1+\sin^2(0)) = 0$$

$$\lim_{x\to k^-} f(x) = (k^3-1)\log_e\left(1+\sin^2(\pi)\right) = (k^3-1)\log_e(1) = 0$$

Since LHL = RHL = f(k) = 0, f(x) is continuous at all integer points

Case II: At points where $x^3 = n \in Z$ but $x \notin \mathbb{Z}$

Since $x \in (-3, 3)$, we have $x^3 \in (-27, 27)$

The total number of integers in the interval (–27, 27) is 26 – (–26) + 1 = 53

Out of these 53 integers, the perfect cubes are {–8, –1, 0, 1, 8} (which correspond to the integer values of x handled in Case I)

For the remaining 53 – 5 = 48 points, x is not an integer. Therefore

$\{x\} \neq 0$, which means $\sin^2(\pi\{x\}) \neq 0$, and thus $\log_e(1+\sin^2(\pi\{x\})) \neq 0$

At $x = n^{1/3}$, the term $[x^3]$ jumps from n – 1 (on the left) to n (on the right), making LHL ≠ RHL

Thus, f(x) is discontinuous at all these 48 points. |A| = 48

The function g is given by: $g(x) = x^3\sin^2\left(\pi\log_e(1+\{x\})\right)$

The only potential points of discontinuity for g(x) are where {x} is discontinuous, i.e., at the integers $x \in \{-2, -1, 0, 1, 2\}$

Let's analyze an arbitrary integer x = k

$$g(k) = k^3\sin^2(\pi\log_e(1+0)) = 0$$

$$\lim_{x\to k^+} g(x) = k^3\sin^2(\pi\log_e(1)) = 0$$

$\lim_{x \to k^-} g(x) = k^3 \sin^2(\pi \log_e(2))$

For g(x) to be continuous at x = k, we require LHL = RHL

$k^3 \sin^2(\pi \log_e(2)) = 0$

Since $\sin^2(\pi \log_e(2)) \neq 0$, this condition is satisfied only when $k^3 = 0 \Rightarrow k = 0$

At x = 0, g(x) is continuous

At $x \in \{-2, -1, 1, 2\}$, g(x) is discontinuous

Thus, the set B = {–2, –1, 1, 2}, which gives $|B| = 4$

Since they share no elements in common, they are disjoint sets $|A \cap B| = 0$

$|A| + 2|B| - |A \cap B| = 48 + 2(4) - 0 = 48 + 8 = 56$

SECTION – 4 (MAXIMUM MARKS: 8)

- This section contains **TWO (02)** question stems.
- This section contains **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If **ONLY** the correct numerical value is entered in the designated place;

Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 15 and 16

Consider the curve C_1 given by

$$y = e^{-x} \quad \text{for } x \in [0, 10\pi]$$

and the curve C_2 given by

$$y = e^{-x}(\sin x + \cos x) \quad \text{for } x \in [0, 10\pi]$$

Let n be the total number of points of intersection of the curves C_1 and C_2

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n \in [0, 10\pi]$ are the x-coordinates of the points of intersection of the curves C_1 and C_2 such that

$$\alpha_1 < \alpha_2 < \ldots < \alpha_n$$

15. The value of n is ________.

Ans. 11

Sol. To find the points of intersection $\quad e^{-x} = e^{-x}(\sin x + \cos x)$

$1 = \sin x + \cos x$

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$x - \frac{\pi}{4} = 2k\pi + \frac{\pi}{4} \Rightarrow x = 2k\pi + \frac{\pi}{2} \text{ or } \quad x - \frac{\pi}{4} = 2k\pi - \frac{\pi}{4} \Rightarrow x = 2k\pi \quad \text{where } k \in \mathbb{Z}$$

Now, we find the values of x that lie within the given domain $x \in [0, 10\pi]$

From $x = 2k\pi$: $x = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$ (6 points)

From $x = 2k\pi + \frac{\pi}{2}$: $\quad x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$ (5 points)

$$\alpha_1 = 0, \alpha_2 = \frac{\pi}{2}, \alpha_3 = 2\pi, \alpha_4 = \frac{5\pi}{2}, \ldots, \alpha_{11} = 10\pi$$

Thus, the total number of intersection points is $n = 11$

16. Let β be the area of the region enclosed between the curves C_1, C_2 and the lines $x = \alpha_1$ and $x = \alpha_4$. Then the value of $-\frac{1}{\pi}\log_e\left(\beta - 2e^{-\frac{\pi}{2}}\right)$ is ________.

Ans. 2.5

Sol. $\alpha_1 = 0$ and $\alpha_4 = \frac{5\pi}{2}$

The enclosed area β is given by

$$\beta = \int_0^{\frac{5\pi}{2}} |y_{C_1} - y_{C_2}|dx = \int_0^{\frac{5\pi}{2}} e^{-x}|1 - \sin x - \cos x|dx$$

First, let's determine the indefinite integral without absolute values

$$I = \int e^{-x}(\sin x + \cos x - 1)dx = F(x)$$

$$F(x) = e^{-x}(1 - \cos x)$$

Since the expression inside the absolute value changes sign only at the roots $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, the total area β is the sum of the absolute differences of F(x) at consecutive boundaries

$$\beta = |F(\alpha_2) - F(\alpha_1)| + |F(\alpha_3) - F(\alpha_2)| + |F(\alpha_4) - F(\alpha_3)|$$

Substituting these values into the area formula

$$\beta = \left|e^{-\frac{\pi}{2}} - 0\right| + \left|0 - e^{-\frac{\pi}{2}}\right| + \left|e^{-\frac{5\pi}{2}} - 0\right|$$

$$\beta = e^{-\frac{\pi}{2}} + e^{-\frac{\pi}{2}} + e^{-\frac{5\pi}{2}} = 2e^{-\frac{\pi}{2}} + e^{-\frac{5\pi}{2}}$$

$$\beta - 2e^{-\frac{\pi}{2}} = e^{-\frac{5\pi}{2}}$$

$$\log_e\left(\beta - 2e^{-\frac{\pi}{2}}\right) = \log_e\left(e^{-\frac{5\pi}{2}}\right) = -\frac{5\pi}{2}$$

Question Stem for Question Nos. 17 and 18

Consider the ellipses given by:

$$x^2 + 4y^2 = 1 \text{ and } 4x^2 + y^2 = 1$$

17. Let P be the point in the first quadrant where the given ellipses intersect. If θ is the acute angle between the tangents to the given ellipses at the point P, then the value of $4\tan\theta$ is _______.

Ans. 7.5

Sol. Given the equations of the two ellipses:

$x^2 + 4y^2 = 1$ and $4x^2 + y^2 = 1$

Subtracting equation (1) from equation (2):

$(4x^2 + y^2) - (x^2 + 4y^2) = 1 - 1$

$3x^2 - 3y^2 = 0 \quad \Rightarrow x^2 = y^2$

Since the point P lies in the first quadrant, both x > 0 and y > 0, so we have x = y.

$x^2 + 4x^2 = 1 \quad \Rightarrow 5x^2 = 1 \quad \Rightarrow x = \frac{1}{\sqrt{5}}$

Since y = x, the coordinates of point P are: $P = \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

Differentiating both equations with respect to x to find the slopes (m_1 and m_2):

For the first ellipse ($x^2 + 4y^2 = 1$): $2x + 8y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{4y}$

At $P\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, the slope m_1 is: $m_1 = -\frac{\frac{1}{\sqrt{5}}}{4\left(\frac{1}{\sqrt{5}}\right)} = -\frac{1}{4}$

For the second ellipse ($4x^2 + y^2 = 1$): $8x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{4x}{y}$

At $P\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, the slope m_2 is: $m_2 = -\frac{4\left(\frac{1}{\sqrt{5}}\right)}{\frac{1}{\sqrt{5}}} = -4$

The angle θ between two lines with slopes m_1 and m_2 is given by:

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \Rightarrow \quad \tan\theta = \left|\frac{-\frac{1}{4} - (-4)}{1 + \left(-\frac{1}{4}\right)(-4)}\right| = \left|\frac{-\frac{1}{4} + 4}{1+1}\right| = \left|\frac{\frac{15}{4}}{2}\right| = \frac{15}{8}$$

$$4\tan\theta = 4 \times \frac{15}{8} = \frac{15}{2} = 7.5$$

18. If α is the area of the common region that lies inside both the given ellipses, then the value of $\cot\alpha$ is ________.

Ans. 0.75

Sol. The given ellipses are highly symmetrical across all four quadrants and about the line $y = x$. We can convert their equations into polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$) to easily integrate the bounded area.

First Ellipse : $r^2\cos^2\theta + 4r^2\sin^2\theta = 1 \quad \Rightarrow r_1^2 = \dfrac{1}{1+3\sin^2\theta}$

Second Ellipse: $4r^2\cos^2\theta + r^2\sin^2\theta = 1 \quad \Rightarrow r_2^2 = \dfrac{1}{4-3\sin^2\theta}$

In the first quadrant, the two curves intersect at $\theta = \dfrac{\pi}{4}$. Due to symmetry, the total area α is 8 times the area bounded from $\theta = 0$ to $\theta = \dfrac{\pi}{4}$ by the inner curve (r_2) :

$$\theta = 8 \times \left(\frac{1}{2}\int_0^{\pi/4} r_2^2\, d\theta\right) = 4\int_0^{\pi/4} \frac{1}{4-3\sin^2\theta}d\theta$$

$$\alpha = 4\int_0^{\pi/4} \frac{\sec^2\theta}{4\sec^2\theta - 3\tan^2\theta}d\theta \Rightarrow \alpha = 4\int_0^{\pi/4} \frac{\sec^2\theta}{4(1+\tan^2\theta) - 3\tan^2\theta}d\theta = 4\int_0^{\pi/4} \frac{\sec^2\theta}{4+\tan^2\theta}d\theta$$

Let $t = \tan\theta \quad \Rightarrow dt = \sec^2\theta\, d\theta$

$$\alpha = 4\int_0^1 \frac{dt}{4+t^2} = 4\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)\right]_0^1 \Rightarrow \quad \alpha = 2\tan^{-1}\left(\frac{1}{2}\right)$$

Let $\beta = \tan^{-1}\left(\dfrac{1}{2}\right)$. Thus, $\alpha = 2\beta$.

$$\tan\alpha = \tan(2\beta) = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

$$\cot\alpha = \frac{1}{\tan\alpha} = \frac{3}{4} = 0.75$$