JEE Main January 2023

Question Paper With Text Solution

31 January | Shift-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JANUARY 2023 | 31TH JANUARY SHIFT-1

SECTION – A

Question ID : 366694607

1. Let $y = f(x) = \sin^3\left(\frac{\pi}{3}\left(\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\right)\right)\right)$. Then, at $x = 1$,

माना $y = f(x) = \sin^3\left(\frac{\pi}{3}\left(\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{\frac{3}{2}}\right)\right)\right)$ है। तो $x = 1$ पर

(1) $\sqrt{2}y'-3\pi^2y=0$ (2) $y'+3\pi^2y=0$

(3) $2y'+3\pi^2y=0$ (4) $2y'+\sqrt{3}\pi^2y=0$

Ans. Official Answer NTA (3)

Sol. $Y = f(x) = \sin^3\left(\frac{\pi}{3}\left(\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)^{3/2}\right)\right)\right)$

$$\frac{dy}{dx} = 3\sin^2\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x^2+1\right)\right)^{3/2}\right)\times\frac{\pi}{3}\left(-\sin\left(\frac{\pi}{3\sqrt{2}}\left(-4x^3+5x+1\right)\right)\right)^{3/2}$$

$$\left(\cos\left(\frac{\pi}{3}\cos\left(\frac{\pi}{3\sqrt{2}}(-4x^3+5x^2+1)^{3/2}\right)\times\frac{\pi}{3\sqrt{2}}(3/2(-4x^3+5x^2+1)^{1/2})(-12x^2+10x\right)\right)$$

at x = 1

$$y = f(1) = \sin^3\left(\frac{\pi}{3}\left(\cos\left(\frac{\pi}{3\sqrt{2}}.2\sqrt{2}\right)\right)\right)$$

$$= \sin^3\left(\frac{\pi}{3}(-1/2)\right)$$

$$= \sin^3\left(-\frac{\pi}{6}\right) = \left(-\frac{1}{2}\right)^3 = -\frac{1}{8}$$

$$\left(\frac{dy}{dx}\right)_{x=1} = 3\left(-\frac{1}{2}\right)^2\cdot\left(-\frac{\pi}{3}\right)\left(\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)\cdot\frac{\pi}{3\sqrt{2}}\left(\frac{3}{2}\sqrt{2}\right)(-2)$$

$$= \frac{3}{4}\cdot\frac{\pi}{3}\cdot\frac{3}{4} = \frac{3\pi^2}{16}$$

Now (3) $2y' + 3\pi^2 y = 2\left(\frac{3\pi^2}{16}\right) + 3\pi^2\left(-\frac{1}{8}\right) = \frac{3\pi^2}{8} - \frac{3\pi^2}{8} = 0$

Question ID : 366694617

2. Let $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$, and $\vec{b}$ and $\vec{c}$ be two non zero vectors such that $|\vec{a}+\vec{b}+\vec{c}| = |\vec{a}+\vec{b}-\vec{c}|$ and $\vec{b}\cdot\vec{c} = 0$. Consider the following two statements :

(A) $|\vec{a}+\lambda\vec{c}| \geq |\vec{a}|$ for all $\lambda \in R$.

(B) $\vec{a}$ and $\vec{c}$ are always parallel.

Then,

(1) bot (A) and (B) are correct

(2) only (A) is correct

(3) only (B) is correct

(4) neither (A) nor (B) is correct.

माना $\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$ तथा $\vec{b}$ और $\vec{c}$ दो शून्येतर सदिश हैं जिनके लिए $|\vec{a}+\vec{b}+\vec{c}| = |\vec{a}+\vec{b}-\vec{c}|$ तथा $\vec{b}\cdot\vec{c} = 0$ है। निम्न दो कथनों का विचार कीजिए :

(A) $\lambda \in R$ के लिए $|\vec{a}+\lambda\vec{c}| \geq |\vec{a}|$

(B) $\vec{a}$ तथा $\vec{c}$ सदा समांतर है

तो,

(1) (A) तथा (B) सही है

(2) केवल (A) सही है

(3) केवल (B) सही है

(4) न तो (A) न ही (B) सही है

Ans. Official Answer NTA (2)

Sol. $\left|\vec{a}+\vec{b}+\vec{c}\right|^2 = \left|\vec{a}+\vec{b}-\vec{c}\right|^2$

$2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a} = 2\vec{a}.\vec{b} - 2\vec{b}.\vec{c} - 2\vec{c}.\vec{a}$

$4\vec{a}.\vec{c} = 0$

B is incorrect.

$\left|\vec{a}+\lambda\vec{c}\right|^2 \geq \left|\vec{a}\right|^2$

$\lambda^2 c^2 \geq 0$

True $\forall \lambda \in R$ (A) is correct.

Question ID : 366694615

3. Let $y = f(x)$ represent a paralbola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $y = -\frac{1}{2}$. Then $S = \left\{x \in R : \tan^{-1}\left(\sqrt{f(x)}\right) + \sin^{-1}\left(\sqrt{f(x)+1}\right) = \frac{\pi}{2}\right\}$:

(1) is an infinite set
(2) is an empty set
(3) contains exactly two elements
(4) contains exactly one element

माना $y = f(x)$ नाभि $\left(-\frac{1}{2}, 0\right)$ तथा नियता $y = -\frac{1}{2}$ के परवलय को निरूपित करता है। तो $S = \left\{x \in R : \tan^{-1}\left(\sqrt{f(x)}\right) + \sin^{-1}\left(\sqrt{f(x)+1}\right) = \frac{\pi}{2}\right\}$:

(1) एक अपरिमित समुच्चय है
(2) एक रिक्त समुच्चय है
(3) में ठीक दो अवयव है
(4) में केवल एक अवयव है

Ans. Official Answer NTA (3)

Sol. Equation of parabola $\sqrt{\left(x+\frac{1}{2}\right)^2 + y^2} = \left|y + \frac{1}{2}\right|$

$$x^2 + \frac{1}{4} + x + y^2 = y^2 + \frac{1}{4} + y$$

$x^2 + x + y = f(x)$

Now $\tan^{-1}\sqrt{f(x)} + \sin^{-1}\sqrt{f(x)+1} = \frac{\pi}{2}$

$$\cos^{-1}\frac{1}{\sqrt{1+f(x)}} + \sin^{-1}\sqrt{f(x)+1} = \frac{\pi}{2}$$

So $\sqrt{f(x)+1} = \frac{1}{\sqrt{1+f(x)}}$

$f(x) + 1 = 1$

$f(x) = 0 \Rightarrow x = 0, -1$

Question ID : 366694606

4. If the sum and product of four positive consecutive terms of a G.P., are 126 and 1296, respectively, then the sum of common ratios of all such GPs is

यदि एक G.P. के चार धनात्मक क्रमागत पदों के योग तथा गुणनफल क्रमशः 126 तथा 1296 है, तो ऐसी सभी G.P. के सार्व अनुपातों का योग है :

(1) 14 (2) $\frac{9}{2}$ (3) 3 (4) 7

Ans. Official Answer NTA (4)

Sol. a, ar, ar^2, ar^3 $(a, r > 0)$

$a^4r^6 = 1296$

$a^2r^3 = 36$

$a = \frac{6}{r^{3/2}}$

$a + ar + ar^2 + ar^3 = 126$

$\frac{1}{r^{3/2}} + \frac{r}{r^{3/2}} + \frac{r^2}{r^{3/2}} + \frac{r^3}{r^{3/2}} = \frac{126}{6} = 21$

$(r^{-3/2} + r^{3/2}) + (r^{1/2} + r^{-1/2}) = 21$

$r^{1/2} + r^{-1/2} = A$

$r^{-3/2} + r^{3/2} + 3A = A^3$

$A^3 - 3A + A = 21$

$A^3 - 2A = 21$

$A = 3$

$\sqrt{r} + \frac{1}{\sqrt{r}} = 3$

$r + 1 = 3\sqrt{r}$

$r^2 + 2r + 1 = 9r$

$r^2 - 7r + 1 = 0$

Question ID : 366694612

5. Let a differentiable function f satisfy $f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}, x \geq 3$. Then $12f(8)$ is equal to :

माना एक अवकलनीय फलन f, $f(x) + \int_3^x \frac{f(t)}{t} dt = \sqrt{x+1}, x \geq 3$ को संतुष्ट करता है। तो $12f(8)$ बराबर है :

(1) 1 (2) 34 (3) 19 (4) 17

Ans. Official Answer NTA (4)

Sol. $f'(x)+\frac{f(x)}{x}=\frac{1}{2\sqrt{x+1}}$

$\frac{dy}{dx}+\frac{y}{x}=\frac{1}{2\sqrt{x+1}}$

$I.F.=e^{\int\frac{1}{x}dx}=x$

$y.x=\int x\frac{1}{2\sqrt{x+1}}dx+c$ put $x + 1 = t^2$, $dx = 2tdt$

$y.x=\frac{1}{2}\int\frac{(t^2-1)2tdt}{t}+c$

$y.x=\frac{1}{2}\left(\frac{2}{3}t^3-2t\right)+c$

$y.x=\frac{\left(\sqrt{x+1}\right)^3}{3}-\sqrt{x+1}+c$

now $f(3)=2$

$6=\frac{(2)^3}{3}-2+c$

$8-\frac{8}{3}=c$

$\frac{16}{3}=c$

$\therefore f(x)=\frac{1}{x}\left(\frac{\left(\sqrt{x+1}\right)^3}{3}-\left(\sqrt{x+1}\right)+\frac{16}{3}\right)$

$\therefore 12f(8)=\frac{12}{8}\left(\frac{27}{3}-3+\frac{16}{3}\right)$

$=\frac{3}{2}\left(6+\frac{16}{3}\right)=17$

Question ID : 366694614

6. If the maximum distance of normal to the ellipse $\frac{x^2}{4}+\frac{y^2}{b^2}=1$, $b<2$, from the origin is 1, then the eccentricity of the ellipse is :

यदि दीर्घवृत्त $\frac{x^2}{4}+\frac{y^2}{b^2}=1$, $b<2$, के अभिलंब की मूलबिंदु से अधिकतम दूरी 1 है, तो दीर्घवृत्त की उत्केन्द्रता है :

(1) $\frac{\sqrt{3}}{4}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$

Ans. Official Answer NTA (2)

Sol. Equation of normal is

$2x \sec\theta - by \operatorname{cosec}\theta = 4 - b^2$

Distance from (0, 0) $= \frac{4-b^2}{\sqrt{4\sec^2\theta + b^2 \operatorname{cosec}^2\theta}}$

Distance is maximum if

$4\sec^2\theta + b^2 \operatorname{cosec}^2\theta$ is minimum

$\Rightarrow \tan^2\theta = \frac{b}{2}$

$\Rightarrow \frac{4-b^2}{\sqrt{4\cdot\frac{b+2}{2}+b^2\cdot\frac{b+2}{b}}}=1$

$\Rightarrow 4-b^2 = b+2 \Rightarrow \beta = 1 \Rightarrow e = \frac{\sqrt{3}}{2}$

Question ID : 366694618

7. A bag contains 6 balls. Two balls are drawn from it at random and both are found to be black. The probability that the bag contains at least 5 black balls is

एक थैले में 6 गेंद हैं। इसमे से यादृच्छया दो गेंद निकाली जाती है तथा दोनों काली पायी जाती है। थैले में कम से कम 5 काली गेंद होने की प्रायिकता है :

(1) $\frac{5}{6}$ (2) $\frac{5}{7}$ (3) $\frac{3}{7}$ (4) $\frac{2}{7}$

Ans. Official Answer NTA (2)

Sol. $\dfrac{{}^5C_2 + {}^6C_2}{{}^2C_2 + {}^3C_2 + {}^4C_2 + {}^5C_2 + {}^8C_2} = \dfrac{10+15}{1+3+6+10+15}$

$= \dfrac{25}{35} = \dfrac{5}{7}$

Question ID : 366694604

8. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{pmatrix}$. Then the sum of the diagonal elements of the matrix $(A+I)^{11}$ is equal to :

माना $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{pmatrix}$ है। तो आव्यूह $(A+I)^{11}$ के विकर्ण के अवयवों का योग है :

(1) 6144 (2) 4094 (3) 2050 (4) 4097

Ans. Official Answer NTA (4)

Sol. $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 12 & -3 \end{bmatrix} = A$

Now $(A+I)^{11} = {}^{11}C_0 A^{11} + {}^{11}C_1 + {}^{11}C_1 A^{10} + \ldots\ldots\ldots + {}^{11}C_{11} I$

$= A({}^{11}C_0 + {}^{11}C_1 + \ldots\ldots\ldots + {}^{11}C_{10}) + I$

$= A(2^{11} - 1) + I$

Trace of $(A+I)^{11}$

$= 2^{11} + 4(2^{11} - 1) + 1 + (-3)(2^{11} - 1) + 1$

$= 2 \times 2^{11} + 1$

$= 2^{12} + 1$

Question ID : 366694605

9. For the system of linear equations

$x + y + z = 6$

$\alpha x + \beta y + 7z = 3$

$x + 2y + 3z = 14,$

which of the following is **NOT** true?

(1) If $\alpha = \beta = 7$, then the system has no solution.

(2) If $\alpha = \beta$ and $\alpha \neq 7$, then the system has a unique solution.

(3) There is a unique point (α, β) on the line $x + 2y + 18 = 0$ for which the system has infinitely many solutions.

(4) For every point $(\alpha, \beta) \neq (7, 7)$ on the line $x - 2y + 7 = 0$, the system has infinitely many solutions.

रैखिक समीकरण निकाय

$x + y + z = 6$

$\alpha x + \beta y + 7z = 3$

$x + 2y + 3z = 14,$

के लिए निम्न में से कौन सा सत्य नहीं है ?

(1) यदि $\alpha = \beta = 7$ है, तो निकाय का कोई हल नहीं है

(2) यदि $\alpha = \beta$ तथा $\alpha \neq 7$ है, तो निकाय का केवल एक हल है

(3) रेखा $x + 2y + 18 = 0$ पर केवल एक बिंदु (α, β) है जिसके लिए निकाय के अनंत हल है

(4) रेखा $x - 2y + 7 = 0$ पर प्रत्येक बिंदु $(\alpha, \beta) \neq (7, 7)$ के लिए निकाय के अनंत हल है

Ans. Official Answer NTA (4)

Sol. By equation 1 and 3 $\quad y + 2z = 8$

$y = 8 - 2z$

And $\quad x = -2 + z$

Now putting in equation 2

$\alpha(z - 2) + \beta(-2z + 8) + 7z = 3$

$\Rightarrow (\alpha - 2\beta + 7)z = 2\alpha - 8\beta + 3$

So equations have unique solution if

$\alpha - 2\beta + 7 \neq 0$

And equations have no solution if

$\alpha - 2\beta + 7 = 0$ and $2\alpha - 8\beta + 3 \neq 0$

And equations have infinite solution if

$\alpha - 2\beta + 7 = 0$ and $2\alpha - 8\beta + 3 = 0$

Question ID : 366694602

10. The number of real roots of the equation $\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 6}$, is :

समीकरण $\sqrt{x^2 - 4x + 3} + \sqrt{x^2 - 9} = \sqrt{4x^2 - 14x + 6}$ के वास्तविक मूलों की संख्या है :

(1) 3 (2) 0 (3) 1 (4) 2

Ans. Official Answer NTA (3)

Sol. $\sqrt{(x-1)(x-3)}+\sqrt{(x-3)(x+3)}=\sqrt{2(2x-1)(x-3)}$

$x-3=0$ and $\sqrt{x-1}+\sqrt{x+3}=\sqrt{2(2x-1)}$ .....(1)

Squaring both side; $x-1+x+3+2\sqrt{(x-1)(x+3)}=4x-2$

$2\sqrt{(x-1)(x+3)}=2x-4$

$\sqrt{x^2+3x-x-3}=x-2$ ...... (2)

Squaring both side; $x^2+2x-3=x^2-4x+4$

$6x=7 \Rightarrow x=\frac{7}{6}$

At $x=\frac{7}{6}$ equation (2) positive = negative

Rejected

$\therefore$ real root $x=3$

Question ID : 366694608

11. A wire of length 20 m is to be cut into two pieces. A piece of length l_1 is bent to make a square of area A_1 and the other piece of length l_2 is made into a circle of area A_2. If $2A_1+3A_2$ is minimu then $(\pi l_1):l_2$ is equal to :

20 मीटर लंबी एक तार को दो भागों में काटा गया है। l_1 लंबाई के एक भाग को मोड़ कर A_1 क्षेत्रफल का एक वर्ग बनाया गया है तथा l_2 लंबाई के दूसरे भाग से A_2 क्षेत्रफल का एक वृत्त बनाया गया है। यदि $2A_1+3A_2$ न्यूनतम है, तो $(\pi l_1):l_2$ बराबर है :

(1) 4 : 1 (2) 1 : 6 (3) 6 : 1 (4) 3 : 1

Ans. Official Answer NTA (3)

Sol. $\ell_1+\ell_2=20 \Rightarrow \frac{d\ell_2}{d\ell_1}=-1$

$A_1=\left(\frac{\ell_1}{4}\right)^2$ and $A_2=\pi\left(\frac{\ell_2}{2\pi}\right)^2$

Let $S=2A_1+3A_2=\frac{\ell_1^2}{8}+\frac{\ell_2^2}{4\pi}$

$\frac{ds}{d\ell}=0 \Rightarrow \frac{2\ell_1}{8}+\frac{6\ell_2}{4\pi}\cdot\frac{d\ell_2}{d\ell_1}=0$

$\Rightarrow \frac{\ell_1}{4} = \frac{6\ell_2}{4\pi} \Rightarrow \frac{\pi\ell_1}{\ell_2} = 6$

Question ID : 366694601

12. Let R be a relation on $N \times N$ defined by (a, b) R (c, d) if and only if $ad(b - c) = bc(a - d)$. Then R is

(1) symmetric but neither reflexive nor transitive

(2) symmetric and transitive but not reflexive

(3) reflexive and symmetric but not transitive

(4) transitive but neither reflexive nor symmetric

माना $N \times N$ पर एक संबंध R, (a, b) R (c, d) यदि और केवल यदि $ad(b - c) = bc(a - d)$ है, द्वारा परिभाषित है। तो R :

(1) सममित है परन्तु न तो स्वतुल्य है न ही संक्रामक है

(2) सममित तथा संक्रामक है परन्तु स्वतुल्य नहीं है

(3) स्वतुल्य तथा सममित है परन्तु संक्रामक नहीं है

(4) संक्रामक है परन्तु न तो स्वतुल्य है न ही सममित है

Ans. Official Answer NTA (1)

Sol. $(a, b)\ R\ (c, d) \Leftrightarrow \frac{b-c}{bc} = \frac{a-d}{ad}$

$\frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{a}$

$\frac{1}{a} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$

Relfexive : $(a, b)\ R\ (a, b) \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$ false $\therefore$ not relfexive

Symmetric : $(a, b)\ R\ (c, d) \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c} \Rightarrow \frac{1}{c} - \frac{1}{d} = \frac{1}{b} - \frac{1}{a} \Rightarrow (c, d)\ R\ (a, b)$ $\therefore$ Symmetric

Transitive : $(a, b)\ R\ (c, d) \Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c}$

$(c, d)\ R\ (e, f) \Rightarrow \frac{1}{c} - \frac{1}{d} = \frac{1}{f} - \frac{1}{e}$

$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f} \neq (a, b)\ R\ (e, f)$ $\therefore$ not transitive

Question ID : 366694619

13. If $\sin^{-1}\frac{\alpha}{17}+\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{77}{36}$, $0<\alpha<13$ then $\sin^{-1}(\sin\alpha)+\cos^{-1}(\cos\alpha)$ is equal to

यदि $\sin^{-1}\frac{\alpha}{17}+\cos^{-1}\frac{4}{5}-\tan^{-1}\frac{77}{36}$, $0<\alpha<13$ है, तो $\sin^{-1}(\sin\alpha)+\cos^{-1}(\cos\alpha)$ बराबर है :

(1) 16 (2) 0 (3) π (4) $16-5\pi$

Ans. Official Answer NTA (3)

Sol. $\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$

$$\therefore \sin^{-1}\frac{\alpha}{17}=\tan^{-1}\frac{77}{36}-\tan^{-1}\frac{3}{4}=\tan^{-1}\left(\frac{\frac{77}{36}-\frac{3}{4}}{1+\frac{77}{36}\cdot\frac{3}{4}}\right)$$

$$\sin^{-1}\frac{\alpha}{17}=\tan^{-1}\frac{8}{15}=\sin^{-1}\frac{8}{17}$$

$$\Rightarrow \frac{\alpha}{17}=\frac{8}{17}\Rightarrow \alpha=8$$

$\therefore \sin^{-1}(\sin 8)+\cos^{-1}(\cos 8)$

$= 3\pi - 8 + 8 - 2\pi$

$= \pi$

Question ID : 366694603

14. For all $z\in C$ on the curve $C_1 : |z| = 4$, let the locus of the point $z+\frac{1}{z}$ be the curve C_2. Then :

(1) the curves C_1 and C_2 intersect at 4 points

(2) the curve C_2 lies inside C_1

(3) the curve C_1 lies inside C_2

(4) the curves C_1 and C_2 intersect at 2 points

माना वक्र $C_1 : |z| = 4$ पर सभी $z\in C$ के लिए बिंदु $z+\frac{1}{z}$ का बिंदुपथ वक्र C_2 है, तो :

(1) वक्र C_1 तथा C_2 चार बिंदुओं पर मिलते हैं

(2) वक्र C_2, वक्र C_1 के अन्दर है

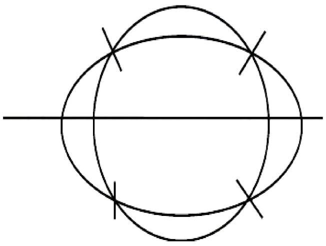
(3) वक्र C_1, वक्र C_2 के अन्दर है

(4) वक्र C_1 तथा C_2 दो बिंदुओं पर मिलते है

Ans. Official Answer NTA (1)

Sol. $|z| = 4$ is a circle $x^2 + y^2 = 16$ and $z+\frac{1}{z}$ is an ellipse $\frac{x^2}{\left(\frac{17}{4}\right)^2}+\frac{y^2}{\left(\frac{15}{4}\right)^2}=1$

$\therefore$ the curves C_1 and C_2 intersect at 4 points



Question ID : 366694611

15. If the domain of the function $f(x)=\frac{[x]}{1+x^2}$, where [x] is greatest integer $\le x$, is [2,6), then its range is

यदि फलन $f(x)=\frac{[x]}{1+x^2}$ जहाँ [x] महत्तम पूर्णांक $\le x$ है का प्रांत [2,6) है, तो इसका परिसर है

(1) $\left(\frac{5}{26},\frac{2}{5}\right]-\left\{\frac{9}{29},\frac{27}{109},\frac{18}{89},\frac{9}{53}\right\}$

(2) $\left(\frac{5}{26},\frac{2}{5}\right]$

(3) $\left(\frac{5}{37},\frac{2}{5}\right]$

(4) $\left(\frac{5}{37},\frac{2}{5}\right]-\left\{\frac{9}{29},\frac{27}{109},\frac{18}{89},\frac{9}{53}\right\}$

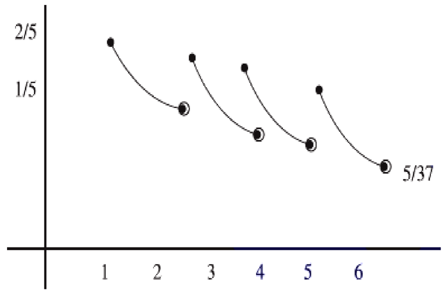
Ans. Official Answer NTA (3)

Sol. $f(x)=\frac{2}{1+x^2}$ $\quad x \in [2, 3)$

$f(x)=\frac{3}{1+x^2}$ $\quad x \in [3, 4)$

$f(x)=\frac{4}{1+x^2}$ $\quad x \in [4, 5)$

$f(x)=\frac{5}{1+x^2}$ $\quad x \in [5, 6)$



$\left(\frac{5}{37}, \frac{2}{5}\right]$

Question ID : 366694620

16. (S1) $(p \Rightarrow q) \vee (p \wedge (\sim q))$ is a tautology

(S2) $((\sim p) \Rightarrow (\sim q)) \wedge ((\sim p) \vee q)$ is a contradiction.

Then

(1) both (S1) and (S2) are correct.

(2) both (S1) and (S2) are wrong

(3) only (S1) is correct

(4) only (S2) is correct

(S1) $(p \Rightarrow q) \vee (p \wedge (\sim q))$ एक पुनरूक्ति है

(S2) $((\sim p) \Rightarrow (\sim q)) \wedge ((\sim p) \vee q)$ एक विरोधोक्ति है

तो

(1) दोनों (S1) तथा (S2) सही है

(2) दोनों (S1) तथा (S2) गलत है

(3) केवल (S1) सही है

(4) केवल (S2) सही है

Ans. Official Answer NTA (1)

Sol.

p	q	$p\Rightarrow q$	$\sim q$	$p\wedge\sim q$	$(p\Rightarrow q)\vee(p\wedge\sim q)$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	T	F	F	T
F	F	T	T	F	T

$\sim p$	$\sim q$	$\sim p\Rightarrow\sim q$	$\sim p\vee q$	$((\sim p)\Rightarrow(\sim q))\wedge(\sim p)\vee q)$
F	F	T	T	T
F	T	T	F	F
T	F	F	T	F
T	T	T	T	T

Question ID : 366694613

17. Let a circle C_1 be obtained on rolling the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ upwards 4 units on the tangent T to it at the point (3,2). Let C_2 be the image of C_1 in T. Let A and B be the centres of circles C_1 and C_2 respectively, and M and N be respectively the feet of perpendiculars drawn from A and B on the x-axis. Then the area of the trapezium AMNB is :

माना वृत्त $x^2 + y^2 - 4x - 6y + 11 = 0$ को इसके बिंदु (3,2) पर स्पर्श रेखा T पर ऊपर की ओर 4 इकाई तक घुमाने पर वृत्त C_1 प्राप्त होता है। माना T में C_1 का प्रतिबिंब C_2 है। माना वृत्तों C_1 तथा C_2 के केन्द्र क्रमशः A तथा B है, और A तथा B से x-अक्ष पर डाले गए लंबों के पाद क्रमशः M तथा N है। तो समलंब AMNB का क्षेत्रफल है :

(1) $2\left(2+\sqrt{2}\right)$ (2) $3+2\sqrt{2}$ (3) $4\left(1+\sqrt{2}\right)$ (4) $2\left(1+\sqrt{2}\right)$

Ans. Official Answer NTA (3)

Sol. $C = (2, 3),\ r = \sqrt{2}$

Centre of G = A = $2 + 4\dfrac{1}{\sqrt{2}}$,

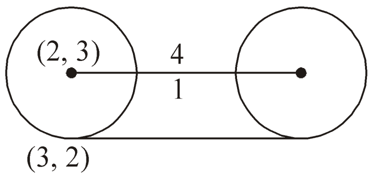
$3+\dfrac{4}{\sqrt{2}} = \left(2+\sqrt{2}, 3+2\sqrt{2}\right)$

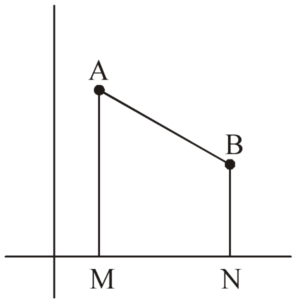
$A\left(2+2\sqrt{2}, 3+2\sqrt{2}\right)$

$B\left(4+2\sqrt{2}, 1+2\sqrt{2}\right)$

$$\frac{x-\left(2+2\sqrt{2}\right)}{1} = \frac{y-\left(3+2\sqrt{2}\right)}{-1} = 2$$

$\therefore$ area of trapezim : $\frac{1}{2}\left(4+4\sqrt{2}\right)2=4\left(1+\sqrt{2}\right)$





Question ID : 366694616

18. Let the shortest distance between the lines $L:\frac{x-5}{-2}=\frac{y-\lambda}{0}=\frac{z+\lambda}{1},\lambda\ge 0$ and $L_1: x+1=y-1=4-z$ be $2\sqrt{6}$. If (α,β,γ) lies on L, then which of the following is **NOT** possible?

माना रेखाओं $L:\frac{x-5}{-2}=\frac{y-\lambda}{0}=\frac{z+\lambda}{1},\lambda\ge 0$ तथा $L_1: x+1=y-1=4-z$ के बीच न्यूनतम दूरी $2\sqrt{6}$ है। यदि (α,β,γ) रेखा L पर है, तो निम्न में से कौन सा संभव मान नहीं है ?

(1) $2\alpha-\gamma=9$ (2) $\alpha+2\gamma=24$ (3) $2\alpha+\gamma=7$ (4) $\alpha-2\gamma=19$

Ans. Official Answer NTA (2)

Sol. $\overrightarrow{b_1}\times\overrightarrow{b_2}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & 1\\ 1 & 1 & -1\end{vmatrix}=-\hat{i}-\hat{j}-2\hat{k}$

$\overrightarrow{a_2}-\overrightarrow{a1}=6\hat{i}+(\lambda-1)\hat{j}+(-\lambda-4)\hat{k}$

$2\sqrt{6}=\left|\frac{-6-\lambda+1+2\lambda+8}{\sqrt{1+1+4}}\right|$

$|\lambda+3|=12\Rightarrow\lambda=9,-15$

$\alpha=-2k+5,\gamma=k-\lambda$ where $k\in R$

$\Rightarrow\alpha+2\gamma=5-2\lambda=-13, 35$

Question ID : 366694609

19. The value of $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(2+3\sin x)}{\sin x(1+\cos x)}dx$ is equal to

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{(2+3\sin x)}{\sin x(1+\cos x)}dx$ का मान बराबर है

(1) $\frac{10}{3}-\sqrt{3}-\log_e\sqrt{3}$

(2) $-2+3\sqrt{3}+\log_e\sqrt{3}$

(3) $\frac{10}{3}-\sqrt{3}+\log_e\sqrt{3}$

(4) $\frac{7}{2}-\sqrt{3}-\log_e\sqrt{3}$

Ans. Official Answer NTA (3)

Sol. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2(1-\cos x)}{\sin x\left(1-\cos^2 x\right)}dx + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{3}{1+\cos x}dx$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{2}{\sin^3 x}dx - 2\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \cot x.\operatorname{cosec}^2 x dx + \frac{3}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sec^2\frac{x}{2}dx$$

$$= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{1+\cot^2 x}.\operatorname{cosec}^2 x dx - 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot x.\operatorname{cosec}^2 x ds + 3\left[\tan\frac{x}{2}\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

Let cot x = 1

$$= -2\int_{\frac{1}{\sqrt{3}}}^{0} \sqrt{1+t^2}dt + 2\int_{\frac{1}{\sqrt{3}}}^{0} tdt + 3\left[1-\frac{1}{\sqrt{3}}\right]$$

$$= 2\left[\frac{t}{2}\sqrt{1+t^2}+\frac{1}{2}\log\left(t+\sqrt{1+t^2}\right)\right]_0^{\frac{1}{\sqrt{3}}} - \left[t^2\right]_0^{\frac{1}{\sqrt{3}}} + 3-\sqrt{3}$$

$= \frac{2}{3} + \log\sqrt{3} - \frac{1}{3} + 3 - \sqrt{3} = \frac{10}{3} - \sqrt{3} + \frac{1}{2}\log 3$

Question ID : 366694610

20. Let $\alpha \in (0,1)$ and $\beta = \log_e(1-\alpha)$. Let $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n}, x \in (0,1)$. Then the integral $\int_0^{\alpha} \frac{t^{50}}{1-t} dt$ is equal to

माना $\alpha \in (0,1)$ तथा $\beta = \log_e(1-\alpha)$ है। माना $P_n(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^n}{n}, x \in (0,1)$ है। तो समाकलन $\int_0^{\alpha} \frac{t^{50}}{1-t} dt$ बराबर है :

(1) $-(\beta + P_{50}(\alpha))$ (2) $\beta + P_{50}(\alpha)$ (3) $\beta - P_{50}(\alpha)$ (4) $P_{50}(\alpha) - \beta$

Ans. Official Answer NTA (1)

Sol. $\int_0^{\alpha} \frac{t^{50} - 1 + 1}{1-t} = -\int_0^{\alpha}(1 + t + ... + t^{49}) + \int_0^{\alpha} \frac{1}{1-t} dt$

$= -\left(\frac{\alpha^{50}}{50} + \frac{\alpha^{49}}{49} + ... + \frac{\alpha^1}{1}\right) + \left(\frac{\ln(1-f)}{-1}\right)_0^{\alpha}$

$= -P_{50}(\alpha) - \ln(1-\alpha)$

$= -P_{50}(\alpha) - \beta$

SECTION – B

Question ID : 366694625

21. Let $a_1, a_2, ..., a_n$ be in A.P. If $a_5 = 2a_7$ and $a_{11} = 18$, then

$12\left(\frac{1}{\sqrt{a_{10}} + \sqrt{a_{11}}} + \frac{1}{\sqrt{a_{11}} + \sqrt{a_{12}}} + ... + \frac{1}{\sqrt{a_{17}} + \sqrt{a_{18}}}\right)$ is equal to ____________.

माना $a_1, a_2, ..., a_n$ A.P में है। यदि $a_5 = 2a_7$ तथा $a_{11} = 18$ है, तो

$$12\left(\frac{1}{\sqrt{a_{10}}+\sqrt{a_{11}}}+\frac{1}{\sqrt{a_{11}}+\sqrt{a_{12}}}+....+\frac{1}{\sqrt{a_{17}}+\sqrt{a_{18}}}\right)$$ बराबर है।

Ans. Official Answer NTA (8)

Sol. Given a + 4d = 2(a + 6d)

a + 8d = 0 .... (1)

and a_{11} = 18 ⇒ a + 10d = 18 .... (2)

From (1) and (2)

–8d + 10d = 18 ⇒ d = 9 and a = –72

a_{10} = a + 9d = –72 + 81 = 9

a_{18} = a + 17d = –72 + 153 = 81

Now

$$12\left[\frac{1}{\sqrt{a_{10}}+\sqrt{a_{11}}}+\frac{1}{\sqrt{a_{11}}+\sqrt{a_{12}}}+........+\frac{1}{\sqrt{a_{17}}+\sqrt{a_{18}}}\right]$$

$$12\left[\frac{\sqrt{a_{10}}-\sqrt{a_{11}}}{-d}+\frac{\sqrt{a_{11}}-\sqrt{a_{12}}}{-d}+........+\frac{\sqrt{a_{17}}-\sqrt{a_{18}}}{-d}\right]$$

$$12\left[\frac{\sqrt{a_{10}}-\sqrt{a_{18}}}{-d}\right]=12\left[\frac{3-9}{-9}\right]=12\times\frac{6}{9}=8$$

Question ID : 366694621

22. Let 5 digit numbers be constructed using the digits 0,2,3,4,7,9 with repetition allowed, and are arranged in ascending ordere with serial numbers. Then the serial number of the number 42923 is _______ .

माना अंकों 0,2,3,4,7,9 के प्रयोग से पुनरावृत्ति के साथ 5 अंकों की संख्याएँ बनाई गई हैं तथा उन्हें क्रम संख्या के साथ आरोही क्रम में व्यवस्थित किया गया है। तो संख्या 42923 की क्रम संख्या है।

Ans. Official Answer NTA (2997)

Sol. $2\underset{6}{+}\underset{6}{+}\underset{6}{+}\underset{6}{+}=1296$

$3\underset{6}{+}\underset{6}{+}\underset{6}{+}\underset{6}{+}=1296$

$40\underset{6}{+}\underset{6}{+}\underset{6}{+}=216$

420 + + = 36
6 6

422 + + = 36
6 6

423 + + = 36
6 6

424 + + = 36
6 6

427 + + = 36
6 6

4290 + = 6
6

429 2 0 = 1
429 2 2 = 1
429 2 3 = 1
= 2997

Question ID : 366694629

23. If the variance of the frequency distribution

x_i	2	3	4	5	6	7	8
Frequency f_i	3	6	16	α	9	5	6

is 3, then α is equal to ____________ .

यदि बारंबारता बंटन

x_i	2	3	4	5	6	7	8
Frequency f_i	3	6	16	α	9	5	6

का प्रसरण 3 है, तो α बराबर है।

Ans. Official Answer NTA (5)

Sol.

x_i	f_i	$d_i = x_i - 5$	$f_i d_i^2$	$f_i d_i$
2	3	–3	27	-9
3	6	–2	24	-12
4	16	–1	16	–16
5	α	0	0	0
6	9	1	9	9
7	5	2	20	10
8	6	3	54	18

$$\sigma_\alpha^2 = \sigma_d^2 = \frac{\Sigma f_i d_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i d_i}{\Sigma f_i}\right)^2$$

$$= \frac{150}{45+\alpha} - 0 = 3$$

$\Rightarrow 150 = 135 + 3\alpha$

$\Rightarrow 3\alpha = 15 \Rightarrow \alpha = 5$

Question ID : 366694623

24. Number of 4-digit numbers that are less than or equal to 2800 and either divisible by 3 or by 11, is equal to ___________ .

4-अंकों की संख्याओं, जो 2800 से कम या इसके बराबर है तथा 3 या 11 से विभाज्य है, की संख्या है।

Ans. Official Answer NTA (710)

Sol. 1000 – 2799

Divisible by 3

$1002 + (n-1)3 = 2799$

n = 600

Divisbible by 11

$$1 - 2799 \rightarrow \left[\frac{2799}{11}\right] = [254] = 254$$

$$1 - 999 = \left[\frac{999}{11}\right] = 90$$

$1000 - 2799 = 254 - 90 = 164$

Divisible by 33

$$1 - 2799 \rightarrow \left[\frac{2799}{33}\right] = 84$$

$$1 - 999 \rightarrow \left[\frac{999}{33}\right] = 30$$

$1000 - 2799 \rightarrow 54$

$\therefore n(3) + n(11) - n(33)$

$600 + 164 - 54 = 710$

Question ID : 366694624

25. Let $\alpha > 0$, be the smallest number such that the expansion of $\left(x^{\frac{2}{3}} + \frac{2}{x^3}\right)^{30}$ has term $\beta x^{-\alpha}$, $\beta \in IN$. Then α is equal to ________ .

माना $\alpha > 0$ न्यूनतम संख्या है, जिसके लिए $\left(x^{\frac{2}{3}} + \frac{2}{x^3}\right)^{30}$ के प्रसार के एक पद में $x^{-\alpha}$ तो α बराबर है ________

$\beta x^{-\alpha}$, $\beta \in IN$

Ans. Official Answer NTA (2)

Sol. $\alpha > 0$, $\left(x^{2/3} + \frac{2}{x^3}\right)^{30}$

$T_{r+1} = {}^{30}C_r \left(x^{2/3}\right)^{30-r} \left(\frac{2}{x^3}\right)^r$

$= {}^{30}C_r x^{20-\frac{2r}{3}-3r}$

$= {}^{30}C_r (2)^r x^{\frac{60-11r}{3}}$ $\quad 0 \le r \le 30$

For $\beta x^{-\alpha}$, $\alpha > 0$ the smallest value of α is 2 (for r = 6)

$T7 = {}^{30}C_6 (2)^6 x^{-2}$

$\alpha = 2$

Question ID : 366694622

26. The remainder on dividing 5^{99} by 11 is ________ .

5^{99} को 11 से विभाजित करने पर शेषफल है।

Ans. Official Answer NTA (9)

Sol. $5^{99} = 5^4.5^{95}$

$= 625\,[5^5]^{19}$

$= 625\,[3125]^{19}$

$= 625\,[2124 + 1]^{19}$

$= 625\,[11k \times 19 + 1]$

$= 625 \times 11k \times 19 + 625$

$= 11k_1 + 616 + 9$

$= 11(k_2) + 9$

Remainder $= 9$

Question ID : 366694627

27. Let $\vec{a}$ and $\vec{b}$ be two vectors such that $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = \sqrt{6}$ and $|\vec{a} \times \vec{b}| = \sqrt{48}$. Then $\left(\vec{a} \cdot \vec{b}\right)^2$ is equal to ___________ .

माना सदिश $\vec{a}$ तथा $\vec{b}$ इस प्रकार है कि $|\vec{a}| = \sqrt{14}$, $|\vec{b}| = \sqrt{6}$ तथा $|\vec{a} \times \vec{b}| = \sqrt{48}$, तो $\left(\vec{a} \cdot \vec{b}\right)^2$ बराबर है।

Ans. Official Answer NTA (36)

Sol. $\left(\vec{a}.\vec{b}\right)^2 + \left(\vec{a} \times \vec{b}\right) = |a|^2 |b|^2$

$\left(\vec{a}.\vec{b}\right)^2 + 48 = 14 \times 6$

$\left(\vec{a}.\vec{b}\right)^2 = 84 - 48 = 36$

Question ID : 366694626

28. Let for $x \in R$

$f(x) = \frac{x + |x|}{2}$ and $g(x) = \begin{cases} x, & x < 0 \\ x^2, & x \geq 0 \end{cases}$.

The area bounded by the curve $y = (fog)(x)$ and the lines $y = 0$, $2y - x = 15$ is equal to ___________ .

माना $x \in R$ के लिए

$f(x) = \frac{x + |x|}{2}$ तथा $g(x) = \begin{cases} x, & x < 0 \\ x^2, & x \geq 0 \end{cases}$.

है। तो वक्र $y = (fog)(x)$ तथा रेखाओं $y = 0$, $2y - x = 15$ से घिरे क्षेत्र का क्षेत्रफल बराबर है।

Ans. Official Answer NTA (72)

Sol. $f(x) = \frac{x + |x|}{2} = \begin{bmatrix} x & x \geq 0 \\ 0 & x < 0 \end{bmatrix}$

$g(x) = \begin{bmatrix} x^2 & x \geq 0 \\ x & x < 0 \end{bmatrix}$

$$fog(x) = f[g(x)] = \begin{cases} g(x) & g(x) \geq 0 \\ 0 & g(x) < 0 \end{cases}$$

$$fog(x) = \begin{cases} x^2 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

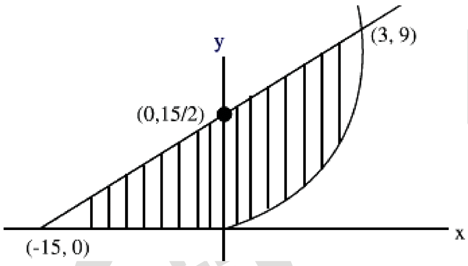
$2y - x = 15$

$$A = \int_0^3 \left(\frac{x+15}{2} - x^2 \right) dx + \frac{1}{2} \times \frac{15}{2} \times 15$$

$$\left. \frac{x^2}{4} + \frac{15x}{2} - \frac{x^3}{3} \right|_0^3 + \frac{225}{4}$$

$$= \frac{9}{4} + \frac{45}{2} - 9 + \frac{225}{4} = \frac{99 - 36 + 225}{4}$$

$$= \frac{288}{4} = 72$$



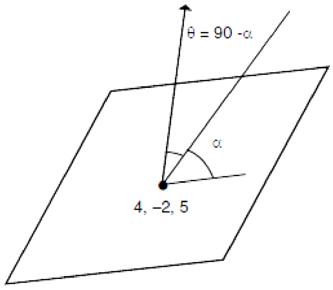
Question ID : 366694630

29. Let θ be the angle between the planes $P_1 : \vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 9$ and $P_2 : \vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 15$. Let L be the line that meets P_2 at the point (4,-2,5) and makes an angle θ with the normal of P_2. If α is the angle between L and P_2, then $(\tan^2\theta)(\cot^2\alpha)$ is equal to ______ .

माना समतलों $P_1 : \vec{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 9$ तथा $P_2 : \vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 15$ के बीच का कोण θ है। माना रेखा L समतल P_2 को बिंदु (4,-2,5) पर मिलती है तथा P_2 के अभिलंब से कोण θ बनाती है। यदि L तथा P_2 के बीच कोण α है तो $(\tan^2\theta)(\cot^2\alpha)$ बराबर है।

Ans. Official Answer NTA (9)

Sol.



$$\cos\theta = \frac{(\hat{i}+\hat{j}+2\hat{k})\cdot(2\hat{i}-\hat{j}+\hat{k})}{6} = \frac{2-1+2}{6} = \frac{1}{2}$$

$\theta = \frac{\pi}{3}$ $\alpha = \frac{\pi}{6}$

$\therefore$ $(\tan^2\theta)(\cot^2\alpha) = (3)(3) = 9$

Question ID : 366694628

30. Let the line $L : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{1}$ intersect the plane $2x+y+3z=16$ at the point P. Let the point Q be the foot of perpendicular from the point R(1,-1,-3) on the line L. If α is the area of triangle PQR, then α^2 is equal to ____________ .

माना रेखा $L : \frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{1}$ तथा समतल $2x+y+3z=16$ का प्रतिच्छेदन बिंदु P है। माना बिंदु R(1,-1,-3) से रेखा L पर लंब का पाद Q है। यदि त्रिभुज PQR का क्षेत्रफल α है तो α^2 बराबर है।

Ans. Official Answer NTA (180)

Sol. Any point on L $((2\lambda+1), (-\lambda-1), (\lambda+3))$

$2(2\lambda+1) + (-\lambda-1) + 3(\lambda+3) = 16$

$6\lambda + 10 = 16 \Rightarrow \lambda = 1$

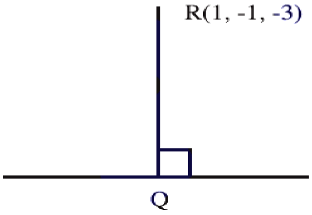
$\therefore$ P = (3, –2, 4)

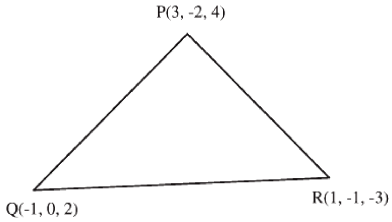
DR of QR = $\langle 2\lambda, -\lambda, \lambda+6\rangle$

DR of L = $\langle 2, -1, 1\rangle$

$4\lambda + \lambda + \lambda + 6 = 0$ $6\lambda + 6 = 0 \Rightarrow \lambda = -1$

Q = (–1, 0, 2)





$\overrightarrow{QR} = 2\hat{i} - \hat{j} - 5\hat{k}$ $\overrightarrow{QP} = 4\hat{i} - 2\hat{j} + 2\hat{k}$

$$\overrightarrow{QR} \times \overrightarrow{QP} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -5 \\ 4 & -2 & 2 \end{vmatrix} = -12i66 - 24\hat{j}$$

$$\alpha = \frac{1}{2} \times \sqrt{144 + 576} \Rightarrow \alpha^2 = \frac{720}{4} = 180$$