JEE Main January 2023

Question Paper With Text Solution

30 January | Shift-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911

Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE MAIN JANUARY 2023 | 30TH JANUARY SHIFT-1

SECTION – A

Question ID : 7155052066

1. If the coefficient of x^{15} in the expansion of $\left(ax^3+\frac{1}{bx^{1/3}}\right)^{15}$ is equal to the coefficient of x^{-15} in the expansion of $\left(ax^{1/3}-\frac{1}{bx^{1/3}}\right)^{15}$, where a and b are positive real numbers, then for each such ordered pair (a, b) :

यदि $\left(ax^3+\frac{1}{bx^{1/3}}\right)^{15}$ के प्रसार में x^{15} का गुणांक, $\left(ax^{1/3}-\frac{1}{bx^{1/3}}\right)^{15}$, के प्रसार में x^{-15} के गुणांक के बराबर है, जहाँ a तथा b घनात्मक संख्याएँ है, तो ऐसे प्रत्येक क्रमित युग्म (a, b) के लिए

(1) a = b (2) ab = 1 (3) a = 3b (4) ab = 3

Ans. Official Answer NTA (2)

Sol. Coefficient of x^{15} in $\left(ax^3+\frac{1}{bx^{1/3}}\right)^{15}$

$$T_{r+1}={}^{15}C_r\left(ax^3\right)^{15-r}\left(\frac{1}{bx^{1/3}}\right)^r$$

$$45-3r-\frac{r}{3}=15$$

$$30=\frac{10r}{3}$$

r = 9

Coefficient of $x^{15}={}^{15}C_9\,a^6\,b^{-9}$

Coefficient of x^{-15} in $\left(ax^{1/3}-\frac{1}{bx^3}\right)^{15}$

$$T_{r+1}={}^{15}C_r\left(ax^{1/3}\right)^{15-r}\left(-\frac{1}{bx^3}\right)^r$$

$$5-\frac{r}{3}-3r=-15$$

$$\frac{10r}{3}=20$$

r = 6

Coefficient = ${}^{15}C_6\, a^9 \times b^{-6}$

$\Rightarrow \frac{a^9}{b^6} = \frac{a^6}{b^9}$

$\Rightarrow a^3b^3 = 1 \Rightarrow ab = 1$

Question ID : 7155052061

2. If $a_n = \frac{-2}{4n^2 - 16n + 15}$, then $a_1 + a_2 + ... + a_{25}$ is equal to :

यदि $a_n = \frac{-2}{4n^2 - 16n + 15}$ है, तो $a_1 + a_2 + ... + a_{25}$ बराबर है :

(1) $\frac{52}{147}$ (2) $\frac{49}{138}$ (3) $\frac{50}{141}$ (4) $\frac{51}{144}$

Ans. Official Answer NTA (3)

Sol. $a_1 + a_2 + ... + a_{25} = \sum_{n=1}^{25} a_n$

$= \Sigma \frac{-2}{4n^2 - 16n + 15} = \Sigma \frac{-2}{(2n-5)(2n-3)}$

$= \sum_{n=1}^{25} \left(\frac{1}{2n-3} - \frac{1}{2n-5} \right)$

$= \left[\left(\frac{1}{-1} - \frac{1}{-3} \right) + \left(\frac{1}{1} - \frac{1}{-1} \right) + \left(\frac{1}{3} - \frac{1}{1} \right) + ... \right.$

$= \frac{1}{2(25)-3} + \frac{1}{3} = \frac{50}{141}$

Question ID : 7155052071

3. Let a unit vector $\widehat{OP}$ make angles α, β, γ with the positive directions of the co-ordinate axes OX, OY, OZ respectively, where $\beta \in \left(0, \frac{\pi}{2}\right)$. If $\widehat{OP}$ is perpendicular to the plane through points (1,2,3), (2,3,4) and (1,5,7), then which one of the following is true?

माना एक इकाई सदिश $\widehat{OP}$ निर्देशांक अक्षों OX, OY, OZ की धनात्मक दिशाओं से क्रमशः कोण α, β, γ बनाते हैं, जहाँ

$\beta \in \left(0, \frac{\pi}{2}\right)$ है। यदि $\widehat{OP}$ बिंदुओं (1,2,3), (2,3,4) तथा (1,5,7) से होकर जाने वाले समतल के लंबवत है, तो निम्न में से कौनसा एक सत्य है ?

(1) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$ (2) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(0, \frac{\pi}{2}\right)$

(3) $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$ (4) $\alpha \in \left(0, \frac{\pi}{2}\right)$ and $\gamma \in \left(\frac{\pi}{2}, \pi\right)$

Ans. Official Answer NTA (3)

Sol. Equation of plane :-

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} = 0$$

$\Rightarrow [x - 1] - 4 [y - 2] + 3 [z - 3] = 0$

$\Rightarrow x - 4y + 3z = 2$

D.R's of normal of plane <1, –4, 3>

D.C's of $\left\langle \pm\frac{1}{\sqrt{26}}, \mp\frac{4}{\sqrt{26}}, \pm\frac{3}{\sqrt{26}} \right\rangle$

$\cos\beta = \frac{4}{\sqrt{26}}$

$\cos\alpha = \frac{-1}{\sqrt{26}}$ $\frac{\pi}{2} < \alpha < \pi$

$\cos\gamma = \frac{-3}{\sqrt{26}}$ $\frac{\pi}{2} < \gamma < \pi$

Question ID : 7155052065

4. Let the solution curve $y = y(x)$ of the differential equation

$$\frac{dy}{dx} - \frac{3x^5 \tan^{-1}(x^3)}{(1+x^6)^{3/2}} y = 2x \exp\left\{\frac{x^3 - \tan^{-1} x^3}{\sqrt{(1+x^6)}}\right\}$$ pass through the origin. Then $y(1)$ is equal to :

माना अवकल समीकरण $\frac{dy}{dx}-\frac{3x^5\tan^{-1}(x^3)}{(1+x^6)^{3/2}}y=2x\exp\left\{\frac{x^3-\tan^{-1}x^3}{\sqrt{(1+x^6)}}\right\}$ का हल वक्र $y=y(x)$ मूल बिंदु से होकर जाता है। तो $y(1)$ बराबर है :

(1) $\exp\left(\frac{\pi-4}{4\sqrt{2}}\right)$ (2) $\exp\left(\frac{1-\pi}{4\sqrt{2}}\right)$ (3) $\exp\left(\frac{4+\pi}{4\sqrt{2}}\right)$ (4) $\exp\left(\frac{4-\pi}{4\sqrt{2}}\right)$

Ans. Official Answer NTA (4)

Sol. $\frac{dy}{dx}-\frac{3x^5\tan^{-1}x^3}{\left(1+x^6\right)^{3/2}}y=2x\exp\left\{\frac{x^3-\tan^{-1}x^3}{\sqrt{1+x^6}}\right\}$

$\text{I.f.}=e^{-\int\frac{3x^5\tan^{-1}x^3}{\left(1+x^6\right)^{3/2}}dx}$

$=e^{-\int\frac{3x^2.x^3\tan^{-1}x^3}{\left(1+x^6\right)^{3/2}}dx}$

Let $\tan^{-1}x^3=t\Rightarrow\frac{3x^2}{1+x^6}dx=dt$

$=e^{-\int\frac{x^3}{\sqrt{1+x^6}}\cdot\frac{3x^2\tan^{-1}x^3}{\left(1+x^6\right)}dx}$

$=e^{-\int\frac{\tan t}{\sec t}.t.dt}$

$=e^{-\int t\sin t.dt}=e^{(t\cos t-\sin t)}$

$=e^{\left(\frac{\tan^{-1}x^3}{\sqrt{1+x^6}}-\frac{x^3}{\sqrt{1+x^6}}\right)}$

Hence solutions of the differential equation is

$y.e^{\frac{\tan^{-1}x^3}{\sqrt{1+x^6}}-\frac{x^3}{\sqrt{1+x^6}}}=\int 2\times e^{\frac{x^3-\tan^{-1}x^3}{\sqrt{1+x^6}}}.e^{\frac{\tan^{-1}x^3-x^3}{\sqrt{1+x^6}}}.dx+C$

$=x^2+C$

Passes through $(0, 0)\Rightarrow C=0$

Hence $y(1)=e^{\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)}$

Question ID : 7155052057

5. Let the system of linear equations

$x + y + kz = 2$

$2x + 3y - z = 1$

$3x + 4y + 2z = k$

have infinitely many solutions. Then the system

$(k + 1) x + (2k - 1) y = 7$

$(2k + 1) x + (k + 5) y = 10$

has :

(1) no solution

(2) unique slution satisfying $x - y = 1$

(3) infinitely many solutions

(4) unique solution satisfying $x + y = 1$

माना रैखिक समीकरण निकाय

$x + y + kz = 2$

$2x + 3y - z = 1$

$3x + 4y + 2z = k$

के अनंत हल है। तो निकाय

$(k + 1) x + (2k - 1) y = 7$

$(2k + 1) x + (k + 5) y = 10$

(1) का कोई हल नहीं हैं

(2) केवल एक हल है जो $x - y = 1$ को संतुष्ट करता है

(3) के अनंत हल है

(4) केवल एक हल है जो $x + y = 1$ को संतुष्ट करता है

Ans. Official Answer NTA (4)

Sol. $$\begin{vmatrix} 1 & 1 & k \\ 2 & 3 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 0$$

$\Rightarrow 1(10) - 1(7) + k(-1) - 0$

$\Rightarrow k = 3$

For k = 3, 2nd system is

$4x + 5y = 7$ ...(1)

and $7x + 8y = 10$ ...(2)

Clearly, they have a unique solution

$(2) - (1) \Rightarrow 3x + 3y = 3$

$\Rightarrow x + y = 1$

Question ID : 7155052058

6. Let $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $d = |A| \neq 0$ and $|A - d(\text{Adj } A)| = 0$. Then

माना $A = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$, $d = |A| \neq 0$ तथा $|A - d(\text{Adj } A)| = 0$ है। तो :

(1) $1 + d^2 = (m + q)^2$

(2) $(1 + d)^2 = (m + q)^2$

(3) $(1 + d)^2 = m^2 + q^2$

(4) $1 + d^2 = m^2 + q^2$

Ans. Official Answer NTA (2)

Sol. $A = \begin{bmatrix} m & n \\ p & q \end{bmatrix}$

$d = mq - np$

$\text{adj}A = \begin{bmatrix} q & -n \\ -p & m \end{bmatrix}$

$|A - d\text{ Adj } A| = 0$

$\begin{bmatrix} m - dq & n + dn \\ p + dp & q - dm \end{bmatrix}$

$(m - dq)(q - dm) - (d + 1)^2 np = 0$

$mq - dm^2 - dq^2 + d^2 mq = (d + 1)^2 np$

$mq(1 + d^2) - d(m^2 + q^2) = (d + 1)^2 (mq - d)$

$mq(1 + d^2) - d(m^2 + q^2) = (d^2 + 2d + 1) mq - (d + 1)^2 d$

$(m + q)^2 = (d + 1)^2$

Question ID : 7155052063

7. The number of points on the curve $y = 54x^5 - 135x^4 - 70x^3 + 180x^2 + 210x$ at which the normal lines are parallel to $x + 90y + 2 = 0$ is :

वक्र $y = 54x^5 - 135x^4 - 70x^3 + 180x^2 + 210x$ पर उन बिंदुओं, जिन पर अभिलंब, रेखा $x + 90y + 2 = 0$ के समांतर

है, की संख्या है :

(1) 3 (2) 0 (3) 4 (4) 2

Ans. Official Answer NTA (3)

Sol. Normal of the line is parallel to line $x + 90y + 2 = 0$

$$m_N = -\frac{1}{90}$$

$$-\left(\frac{dx}{dy}\right)_{(x_1y_1)} = -\frac{1}{90} \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1y_1)} = 90$$

Now,

$$\frac{dy}{dx} = 270x^4 - 540x^3 - 210x^2 + 360x + 210 = 90$$

$$\Rightarrow x = 1, 2, \frac{-2}{3}, \frac{-1}{3}$$

(4) normals

Question ID : 7155052064

8. If [t] denotes the greatest integer $\leq t$, then the value of $\frac{3(e-1)}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$ is :

यदि [t] महत्तम पूर्णांक $\leq t$, $\frac{3(e-1)}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$ का मान है :

(1) $e^8 - 1$ (2) $e^7 - 1$ (3) $e^8 - e$ (4) $e^9 - 1$

Ans. Official Answer NTA (3)

Sol. $$\frac{3(e-1)}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$$

Let $k = \int_1^2 x^2 e^{1+[x^3]}dx$

$x^3 = t$

$3x^2\,dx = dt$

$$k = \frac{1}{3}\int_1^8 e.e^{[t]}dt$$

$$\frac{e}{3}\left[e+e^2+e^3+......+e^7\right]=\frac{e^2}{3}\left(\frac{e^7-1}{e-1}\right)$$

$$\frac{3(e-1)}{e}k=\frac{3(e-1)}{e}\frac{e^2}{3}\left(\frac{e^7-1}{e-1}\right)=e(e^7-1)$$

$= e^8 - e$

Question ID : 7155052059

9. The coefficient of x^{301} in $(1+x)^{500}+x(1+x)^{499}+x^2(1+x)^{498}+.......+x^{500}$ is :

$(1+x)^{500}+x(1+x)^{499}+x^2(1+x)^{498}+.......+x^{500}$ में x^{301} का गुणांक है :

(1) $^{501}C_{302}$ (2) $^{500}C_{300}$ (3) $^{500}C_{301}$ (4) $^{501}C_{200}$

Ans. Official Answer NTA (4)

Sol. $(1+x)^{500}+x(1+x)^{499}+x^2(1+x)^{498}+....+x^{500}$

$$=(1+x)^{500}\left\{\frac{1-\left(\frac{x}{1+x}\right)^{501}}{1-\frac{x}{1+x}}\right\}$$

$$=(1+x)^{500}\frac{\left((1+x)^{501}-x^{501}\right)}{(1+x)^{501}}\cdot(1+x)$$

$=(1+x)^{501}-x^{501}$

Coefficient of x^{301} in $(1+x)^{501}-x^{501}$ is given by $^{501}C_{301}={}^{501}C_{200}$

Question ID : 7155052068

10. If P(h,k) be a point on the parabola $x=4y^2$, which is nearest to the point $Q(0,33)$, then the distance of P from the directrix of the parabola $y^2=4(x+y)$ is equal to :

यदि परवलय $x=4y^2$ पर बिंदु P(h,k), बिंदु $Q(0,33)$ के निकटतम है, तो P की परवलय $y^2=4(x+y)$ की नियता से दूरी बराबर है :

(1) 4 (2) 6 (3) 2 (4) 8

Ans. Official Answer NTA (2)

Sol. $4y^2 = x$

$8y\frac{dy}{dx}=1$

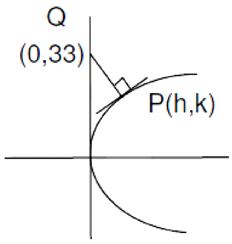
$\frac{dy}{dx}=\frac{1}{8y}$

$m_n=-8K=\frac{K-33}{h}$

$-8K=\frac{K-33}{4k^2}$

$-32k^2=k-33$
$32k^2+k-33=0$
$(32k+33)(k-1)=0$

$k=-\frac{33}{32}$,
$k=1, h=4$
$P(4, 1)$
curve $y^2+4y=4x$
$(y+2)^2=4(x+1)$
dire. $x+1=-1$
$x+2=0$
req. dist. $=|4+2|=6$



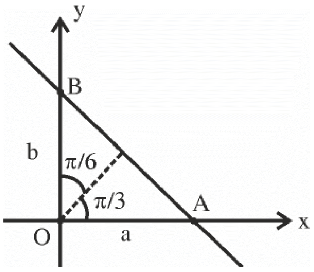
Question ID : 7155052067

11. A straight line cuts off the intercepts OA = a and OB = b on the positive directions of x-axis and y-axis respectively. If the perpendicular from origin O to this line makes an angle of $\frac{\pi}{6}$ with positive direction of y-axis and the area of ΔOAB is $\frac{98}{3}\sqrt{3}$, then $a^2 - b^2$ is equal to :

एक सरल रेखा x-अक्ष तथा y-अक्ष की घनात्मक दिशाओं पर क्रमशः OA = a तथा OB = b अंतःखंड काटती है। यदि मूलबिंदु से इस रेखा पर अभिलंब y-अक्ष की घनात्मक दिशा से $\frac{\pi}{6}$ का कोण बनाता है तथा ΔOAB का क्षेत्रफल $\frac{98}{3}\sqrt{3}$ है, तो $a^2 - b^2$ बराबर है :

(1) 98 (2) $\frac{196}{3}$ (3) 196 (4) $\frac{392}{3}$

Ans. Official Answer NTA (4)

Sol. 

Equation of straight line : $\frac{x}{a}+\frac{y}{b}=1$

Or $x\cos\frac{\pi}{3}+y\sin\frac{\pi}{3}=p$

$\frac{x}{2}+\frac{y\sqrt{3}}{2}=p$

$\frac{x}{3p}+\frac{y}{2p}=1$

Comparing both : $a=2p,\ b=\frac{2p}{\sqrt{3}}$

Now area of $\Delta OAB = \frac{1}{2}\cdot ab=\frac{98}{3}\cdot\sqrt{3}$

$\frac{1}{2}\cdot 2p\cdot\frac{2p}{\sqrt{3}}=\frac{98}{3}\cdot\sqrt{3}$

$p^2 = 49$

$a^2 - b^2 = 4p^2-\frac{4p^2}{3}=\frac{2}{3}4p^2$

$=\frac{8}{3}\cdot 49=\frac{392}{3}$

Question ID : 7155052072

12. If $\vec{a},\vec{b},\vec{c}$ are three non-zero vectors and $\hat{n}$ is a unit vector perpendicular to $\vec{c}$ such that $\vec{a}=\alpha\vec{b}-\hat{n}$, $(\alpha\neq 0)$ and $\vec{b}\cdot\vec{c}=12$, then $|\vec{c}\times(\vec{a}\times\vec{b})|$ is equal to :

यदि $\vec{a},\vec{b},\vec{c}$ तीन शून्येतर सदिश है तथा $\vec{c}$ के लंबवत एक इकाई सदिश $\hat{n}$ है, जिनके लिए $\vec{a}=\alpha\vec{b}-\hat{n}$, $(\alpha\neq 0)$ तथा $\vec{b}\cdot\vec{c}=12$ है, तो $|\vec{c}\times(\vec{a}\times\vec{b})|$ बराबर है :

(1) 12 (2) 9 (3) 6 (4) 15

Ans. Official Answer NTA (1)

Sol. $\vec{n}\cdot\vec{c}=0, |\vec{n}|=1$

$\vec{a}=\alpha\vec{b}-\hat{n}$ & $\vec{b}\cdot\vec{c}=12$

$\vec{a}\cdot\vec{c}=\alpha\vec{b}\cdot\vec{c}-0$

$\vec{a}\cdot\vec{c}=12\alpha$

$\left|\vec{c}\times\left(\vec{a}\times\vec{b}\right)\right|$

$\left|\left(\vec{c}\cdot\vec{b}\right)\vec{a}-\left(\vec{c}\cdot\vec{a}\right)\cdot\vec{b}\right|$

$\left|12\vec{a}-12\alpha\vec{b}\right|$

$12\left|\vec{a}-\alpha\vec{b}\right|$

$12\left|\hat{n}\right|=12$

Question ID : 7155052056

13. The minimum number of elements that must be added to the relation R = {(a,b),(b,c)} on the set {a,b,c} so that it becomes symmetric and transitive is :

समुच्चय {a, b, c} पर संबंध R = {(a,b),(b,c)} में कम से कम कितने अवयव जोड़े जाएं कि संबंध R सममित तथा संक्रामक हो जाए :

(1) 7 (2) 3 (3) 5 (4) 4

Ans. Official Answer NTA (1)

Sol. For Symmetric (a, b), (b, c) $\in$ R

$\Rightarrow$ (b, a), (c, b) $\in$ R

For Transitive (a, b), (b, c) $\in$ R

$\Rightarrow$ (a, c) $\in$ R

Now

1. Symmetric

$\therefore$ (a, c) $\in$ R $\Rightarrow$ (c, a) $\in$ R

2. Transitive

$\therefore$ (a, b), (b, a) $\in$ R

$\Rightarrow$ (a, a) $\in$ R & (b, c), (c, b) $\in$ R

$\Rightarrow$ (b, b) & (c, c) $\in$ R

$\therefore$ Elements to be added

$$\left\{\begin{matrix}(b,a),(c,b),(a,c),(c,a) \\ ,(a,a),(b,b),(c,c)\end{matrix}\right\}$$

Number of elements to be added = 7

Question ID : 7155052069

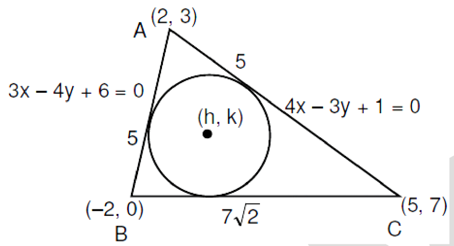
14. Let $y = x + 2$, $4y = 3x + 6$ and $3y = 4x + 1$ be three tangent lines to the circle $(x - h)^2 + (y - k)^2 = r^2$. Then h + k is equal to :

माना $y = x + 2$, $4y = 3x + 6$ तथा $3y = 4x + 1$ वृत्त $(x - h)^2 + (y - k)^2 = r^2$ की तीन स्पर्श रेखाऐं है। तो h + k बराबर है :

(1) 5 (2) $5\sqrt{2}$ (3) 6 (4) $5(1+\sqrt{2})$

Ans. Official Answer NTA (1)

Sol.



$x - y + 2 = 0$

ΔABC is isosceles (AB = AC) so angle bisector & altitude are same then

Equation of angle bisector of angle A is equal to altitude from vertex A

$x + y = \lambda$ (Passes through A)

$2 + 3 = \lambda \Rightarrow \lambda = 5$

$x + y = 5$ (Passes through center)

$h + k = 5$

Question ID : 7155052073

15. If an unbiased die, marked with -2,-1,0,1,2,3 on its faces, is thrown five times, then the probability that the product of the outcomes is positive, is :

यदि एक अनभिनत पासे, जिसके फलकों पर -2,-1,0,1,2,3 लिखा है, को पाँच बार फेंका जाता है, तो फलकों पर प्राप्त संख्याओं का गुणनफल धनात्मक होने की प्रायिकता है :

(1) $\frac{440}{2592}$ (2) $\frac{27}{288}$ (3) $\frac{521}{2592}$ (4) $\frac{881}{2592}$

Ans. Official Answer NTA (3)

Sol. Either all outcomes are positive or any two are negative.

Now, $p = P\,(\text{positive}) = \frac{3}{6} = \frac{1}{2}$

$q = p\,(\text{negative}) = \frac{2}{6} = \frac{1}{3}$

Required probability

$= {}^5C_5\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right)^3 + {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{1}{2}\right)^1$

$= \frac{521}{2592}$

$\therefore$ Option (3) is correct.

Question ID : 7155052074

16. If $\tan 15^\circ + \frac{1}{\tan 75^\circ} + \frac{1}{\tan 105^\circ} + \tan 195^\circ = 2a$, then the value of $\left(a + \frac{1}{a}\right)$ is :

यदि $\tan 15^\circ + \frac{1}{\tan 75^\circ} + \frac{1}{\tan 105^\circ} + \tan 195^\circ = 2a$ है, तो $\left(a + \frac{1}{a}\right)$ का मान है :

(1) $4 - 2\sqrt{3}$ (2) 4 (3) 2 (4) $5 - \frac{3}{2}\sqrt{3}$

Ans. Official Answer NTA (2)

Sol. $\Rightarrow \tan 15^\circ + \tan 15^\circ - \tan 15 + \tan 15 = 2a$

$\Rightarrow a = \tan 15^\circ$

Now $\left(a + \frac{1}{a}\right) = \tan 15^\circ + \frac{1}{\tan 15^\circ}$

$= \frac{1 + \tan^2 15^\circ}{\tan 15^\circ} = \frac{2}{\sin 30^\circ} = 4$

Question ID : 7155052062

17. Suppose $f: R \to (0, \infty)$ be a differentiable function such that $5f(x + y) = f(x) \cdot f(y), \forall x, y \in R$. If $f(3) = 320$, then $\sum_{n=0}^{5} f(n)$ is equal to :

माना एक अवकलनीय फलन $f: R \to (0, \infty)$ के लिए $5f(x + y) = f(x) \cdot f(y), \forall x, y \in R$. है। यदि $f(3) = 320$ तो

$\sum_{n=0}^{5} f(n)$ बराबर है :

(1) 6825 (2) 6525 (3) 6575 (4) 6875

Ans. Official Answer NTA (1)

Sol. $5f(x+y) = f(x) \cdot f(y)$

$5f(0) = f(0)^2 \Rightarrow f(0) = 5$

$5f(x+1) = f(x) \cdot f(1)$

$$\Rightarrow \frac{f(x+1)}{f(x)} = \frac{f(1)}{5}$$

$$\Rightarrow \frac{f(1)}{f(0)} \cdot \frac{f(2)}{f(1)} \cdot \frac{f(3)}{f(2)} = \left(\frac{f(1)}{5}\right)^3$$

$$\Rightarrow \frac{320}{5} = \frac{(f(1))^3}{5^3} \Rightarrow f(1) = 20$$

$\therefore 5f(x+1) = 20 \cdot f(x) \Rightarrow f(x+1) = 4f(x)$

$$\sum_{n=0}^{5} f(n) = 5 + 5.4 + 5.4^2 + 5.4^3 + 5.4^4 + 5.4^5$$

$$= \frac{5\left[4^6 - 1\right]}{3} = 6825$$

Question ID : 7155052070

18. The line l_1 passes through the point (2,6,2) and is perpendicular to the plane $2x + y - 2z = 10$. Then the shortest distance between the line l_1 and the line $\frac{x+1}{2} = \frac{y+4}{-3} = \frac{z}{2}$ is :

रेखा l_1 बिंदु (2,6,2) से होकर जाती है ताथा समतल $2x + y - 2z = 10$ पर लंबवत है। तो l_1 तथा रेखा $\frac{x+1}{2} = \frac{y+4}{-3} = \frac{z}{2}$ के बीच न्यूनतम दूरी है :

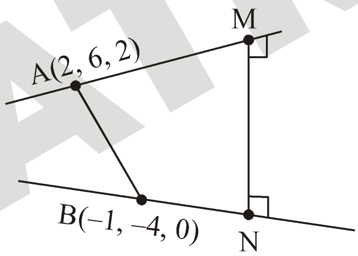
(1) 7 (2) $\frac{13}{3}$ (3) 9 (4) $\frac{19}{3}$

Ans. Official Answer NTA (3)

Sol. Line l_1 is given by $L_1 : \frac{x-2}{2} = \frac{y-6}{1} = \frac{z-2}{-2}$

Given,

$L_3 : \frac{x+1}{2} = \frac{y+4}{-3} = \frac{z}{-2}$



Shortest distance $= \left|\frac{\overrightarrow{AB}.\overrightarrow{MN}}{MN}\right|$

$\overrightarrow{AB} = 3\hat{i} + 10\hat{j} + 2\hat{k}$

$\overrightarrow{MN} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 2 & -3 & 2 \end{vmatrix} = -4\hat{i} - 8\hat{j} - 8\hat{k}$

$\therefore$ $MN = \sqrt{16+64+64} = 12$

$\therefore$ Shortest distnace $= \left|\frac{-12-80-16}{12}\right| = 9$

$\therefore$ Option (3) is correct

Question ID : 7155052075

19. Among the statements :

(S1) $\left((p \vee q) \Rightarrow r\right) \Leftrightarrow (p \Rightarrow r)$

(S2) $\left((p \vee q) \Rightarrow r\right) \Leftrightarrow \left((p \Rightarrow r) \vee (q \Rightarrow r)\right)$

(1) neither (S1) nor (S2) is tautology

(2) only (S1) is a tautology

(3) both (S1) and (S2) are tatologies

(4) only (S2) is a tautology

कथनों

(S1) $\left((p \vee q) \Rightarrow r\right) \Leftrightarrow (p \Rightarrow r)$

(S2) $\left((p \vee q) \Rightarrow r\right) \Leftrightarrow \left((p \Rightarrow r) \vee (q \Rightarrow r)\right)$

में से

(1) न तो (S1) नहीं (S2) पुनरूक्ति है

(2) केवल (S1) पुनरूकि है

(3) (S1) तथा (S2) पुनरूक्ति है

(4) केवल (S2) पुनरूक्ति है

Ans. Official Answer NTA (1)

Sol. If P = F, q = T, r = F

$S_1 : ((F \vee T) \to F) \Leftrightarrow (F \to F) = (T \to F) \Leftrightarrow (T = F) \Leftrightarrow T = F$

$S_2 : ((F \vee T) \to F) \Leftrightarrow (F \Rightarrow F) \vee (T \Rightarrow F) \Leftrightarrow (T \vee F)$

$= F \Leftrightarrow T = F$

Question ID : 7155052060

20. If the solution of the equation $\log_{\cos x} \cot x + 4 \log_{\sin x} \tan x = 1,\ x \in \left(0, \frac{\pi}{2}\right)$, is $\sin^{-1}\left(\frac{\alpha + \sqrt{\beta}}{2}\right)$ where α, β are integers, then $\alpha + \beta$ is equal to :

यदि समीकरण $\log_{\cos x} \cot x + 4 \log_{\sin x} \tan x = 1,\ x \in \left(0, \frac{\pi}{2}\right)$ का हल $\sin^{-1}\left(\frac{\alpha + \sqrt{\beta}}{2}\right)$ हैं, जहाँ α, β पूर्णांक है, तो $\alpha + \beta$ बराबर है :

(1) 6 (2) 3 (3) 4 (4) 5

Ans. Official Answer NTA (3)

Sol. $\log_{\cos x} \cot x + 4 \cdot \log_{\sin x} \tan x = 1$

$$\frac{\log\cos x - \log\sin x}{\log\cos x} + 4\cdot\frac{(\log\sin x - \log\cos x)}{\log\sin x} = 1$$

$(\log \sin x)^2 - 4(\log \sin x)(\log \cos x) + 4(\log \cos x)^2 = 1$

$\log \sin x = 2 \log \cos x$

$\sin^2 x + \sin x - 1 = 0$

$$\sin x = \frac{-1 \pm \sqrt{5}}{2}$$

$\alpha = -1, \beta = 5$

$\alpha + \beta = 4$

SECTION – B

Question ID : 7155052083

21. If $\lambda_1 < \lambda_2$ are two values of λ such that the angle between the planes $P_1 : \vec{r}\left(3\hat{i} - 5\hat{j} + \hat{k}\right) = 7$ and $P_2 : \vec{r}\left(\lambda\hat{i} + \hat{j} - 3\hat{k}\right) = 9$ is $\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$, then the square of the length of perpendicular from the point $(38\lambda_1, 10\lambda_2, 2)$ to the plane P_1 is ________ .

यदि λ के दो मान, जिनके लिए समतलों $P_1 : \vec{r}\left(3\hat{i} - 5\hat{j} + \hat{k}\right) = 7$ तथा $P_2 : \vec{r}\left(\lambda\hat{i} + \hat{j} - 3\hat{k}\right) = 9$ के बीच का कोण $\sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$ है, $\lambda_1, \lambda_2, (\lambda_1 < \lambda_2)$ है, तो बिंदु $(38\lambda_1, 10\lambda_2, 2)$ से समतल P_1 पर लंब की लंबाई का वर्ग बराबर है।

Ans. Official Answer NTA (315)

Sol. $P_1 : \vec{r}\left(3\hat{i} - 5\hat{j} + \hat{k}\right) = 7$ and

$P_2 : \vec{r}\left(\lambda\hat{i} + \hat{j} - 3\hat{k}\right) = 9$

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

$$\cos\theta = \frac{(3\lambda - 5 - 3)}{\sqrt{35}\cdot\sqrt{\lambda^2 + 10}}$$

$$\cos\theta = \frac{(3\lambda - 8)}{\sqrt{35}\cdot\sqrt{\lambda^2 + 10}}$$

$$\therefore \theta = \sin^{-1}\left(\frac{2\sqrt{6}}{5}\right)$$

$$\sin\theta = \frac{2\sqrt{6}}{5}$$

$$\sin^2\theta = \frac{24}{25}$$

$$1 - \cos^2\theta = \frac{24}{25}$$

$$1 - \frac{(3\lambda - 8)^2}{35(\lambda^2 + 10)} = \frac{24}{25}$$

$25 [35 (\lambda^2 + 10) - (3\lambda - 8)^2] = 35\cdot 24 (\lambda^2 + 10)$
$5 [35\lambda^2 + 350 - 9\lambda^2 - 64 + 48\lambda] = 168\lambda^2 + 1680$
$175\lambda^2 + 1750 - 45\lambda^2 - 320 + 240\lambda - 168\lambda^2 - 1680 = 0$
$-38\lambda^2 + 240\lambda - 250 = 0$
$19\lambda^2 - 120\lambda + 125 = 0$

$$\lambda = 5, \frac{25}{19}$$

$$\because \lambda_1 < \lambda_2$$

$$\therefore \lambda_2 = 5, \lambda_1 = \frac{25}{19}$$

then point $\left(38\frac{25}{19}, 10\,52\right) = (50, 50, 2)$

$P_1 : 3x - 5y + z = 7$

$$d = \left|\frac{3\times 50 - 5\times 50 + 1\cdot 2 - 7}{\sqrt{9 + 25 + 1}}\right|$$

$$d = \left|\frac{150 - 250 + 2 - 7}{\sqrt{35}}\right|$$

$$d = \left|\frac{-100 - 5}{\sqrt{35}}\right| = \frac{105}{\sqrt{35}}$$

$$d^2 = \frac{105\times 105}{35}$$

$= 315$

Question ID : 7155052078

22. Let $S = \{1,2,3,4,5,6\}$. Then the number of one-one function $f: S \to P(S)$, where P(S) denote the power set of S, such that $f(n) \subset f(m)$ where $n < m$ is ________ .

माना $S = \{1,2,3,4,5,6\}$ है तो ऐसे एकैकी फलनों $f: S \to P(S)$, जहाँ P(S) समुच्चय S का घात समुच्चय $f(n) \subset f(m)$ है जब कि $n < m$ है, की संख्या है।

Ans. Official Answer NTA (3240)

Sol. Let S = {1, 2, 3, 4, 5, 6}, then the number of one-one functions, f : S•P (S), where P(S) denotes the power set of S, such that f(n) < f(m) where n < m is
n (S) = 6
$P(S) = \{\phi, \{1\}, ..., \{6\}, \{1,2\}, ..., \{5.6\}, ..., \{1,2,3,4,5,6,\}\}$
–64 elements
case – 1
f (6) = S i.e. 1 option,
f (5) = any 5 element subset A of S i.e. 6 options,
f (4) = any 4 element subset B of A i.e. 5 options,
f (3) = any 3 element subset C of B i.e. 4 options,
f (2) = any 2 element subset D of C i.e. 3 options,
f (1) = any 1 element subset E of D or empty subset i.e. 3 options,
Total functions = 1080
Case – 2
f (6) = any 5 element subset A of S i.e. 6 options,
f (5) = any 4 element subset B of A i.e. 5 options,
f? (4) = any 3 element subset C of B i.e. 4 options,
f (3) = any 2 element subset D of C i.e. 3 options,
f? (2) = any 1 element subset E of D i.e. 2 options,
f (1) = empty subset i.e. 1 option
Total functions = 720
Case – 3
f(6)= S
f(5) = any 4 element subset A of' S i.e. 15 options,
f(4) = any 3 element subset B of A i.e. 4 options,
f(3) = any 2 element subset C of B i.e. 3 options,
f(2) = any 1 element subset D of C i.e. 2 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
Case – 4
f(6) = S

f(5) = any 5 element subset A of S i.e. 6 options,
f(4) = any 3 element subset B of A i.e. 10 options,
f(3) = any 2 element subset C of B i.e. 3 options,
f(2) = any 1 element subset D of C i.e. 2 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
Case – 5
f(6) = S
f(5) = any 5 element subset A of S i.e. 6 options,
f(4) = any 4 element subset B of A i.e. 5 options,
f(3) = any 2 element subset C of B i.e. 6 options,
f(2) = any 1 element subset D of C i.e. 2 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
Case – 6
f(6) = S
f(5) = any 5 element subset A of S i.e. 6 options,
f(4) = any 4 element subset B of A i.e. 5 options,
f(3) = any 3 element subset C of B i.e. 4 options,
f(2) = any 1 element subset D of C i.e. 3 options,
f(1) = empty subset i.e. 1 option
Total functions = 360
$\therefore$ Number of such functions = 3240

Question ID : 7155052082

23. Number of 4-digit numbers (the repetition of digits is allowed) which are made using the digits 1,2,3 and 5, and are divisible by 15, is equal to _________ .

अंकों 1,2,3 तथा 5 के प्रयोग से (अंकों की पुनरावृत्ति की अनुमति है) 15 से विभाज्य 4 अंकों की संख्याओं की संख्या है।

Ans. Official Answer NTA (21)

Sol. divisible by 15,
In which last digit should be 5 and sum of digits must be divisible by 3

C–I : $\underline{1}\ \underline{2}\ \underline{1}\ \underline{(5)} \rightarrow \frac{3!}{2!} = 3$

$\underline{2}\ \underline{2}\ \underline{3}\ \underline{(5)} \rightarrow \frac{3!}{2!} = 3$

$\underline{3}\ \underline{3}\ \underline{1}\ \underline{(5)} \rightarrow \frac{3!}{2!} = 3$

$\underline{1}\ \underline{1}\ \underline{5}\ \underline{(5)} \rightarrow \frac{3!}{2!} = 3$

$\underline{2}\ \underline{3}\ \underline{5}\ \underline{(5)} \rightarrow 3! = 6$

$\underline{3}\ \underline{5}\ \underline{5}\ \underline{(5)} \rightarrow \frac{3!}{2!} = 3$

Total = 21

Question ID : 7155052080

24. If the equation of the plane passing through the point (1,1,2) and are perpendicular to the line $x - 3y + 2z - 1 = 0 = 4x - y + z$ is $Ax + By + Cz = 1$, then 140 (C - B + A) is equal to _______ .

यदि बिंदु (1,1,2) से होकर जाने वाला तथा रेखा $x - 3y + 2z - 1 = 0 = 4x - y + z$ के लंबवत् समतल का समीकरण $Ax + By + Cz = 1$ है, तो 140 (C - B + A) बराबर है।

Ans. Official Answer NTA (15)

Sol. $x - 3y + 2z - 1 = 0$

$4x - y + z = 0$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 4 & -1 & 1 \end{vmatrix}$$

$= -\hat{i} + 7\hat{j} + 11\hat{k}$

D.R. of normal to the plane is –1, 7, 11

Equation of plane

$-1(x - 1) + 7(y - 1) + 11(z - 2) = 0$

$-x + 7y + 11z = 28$

$-\frac{1}{28}x + \frac{7}{28}y + \frac{11}{28}z = 1$

$Ax + By + Cz = 1$

$140(C - B + A) = 140\left(\frac{11}{28} - \frac{7}{28} - \frac{1}{28}\right)$

$= 140 \times \frac{3}{28} = 15$

Question ID : 7155052081

25. $\lim_{x \to 0} \frac{48}{x^4} \int_0^x \frac{t^3}{t^6 + 1} dt$ is equal to _____ .

$\lim_{x\to 0}\frac{48}{x^4}\int_0^x \frac{t^3}{t^6+1}dt$ बराबर है।

Ans. Official Answer NTA (12)

Sol. $\lim_{x\to 0}\frac{48}{x^4}\int_0^x \frac{t^3}{t^6+1}dt$

$$= \lim_{x\to 0}\frac{48\int_0^x \frac{t^3}{t^6+1}\cdot dt}{x^4}$$

$$= \lim_{x\to 0}\frac{48\times \frac{x^3}{x^6+1}}{4x^3} = \lim_{x\to 0}\frac{12}{x^6+1} = 12$$

Question ID : 7155052085

26. Let α be the area of the larger region bounded by the curve $y^2 = 8x$ and the lines $y = x$ and $x = 2$, which lies in the first quadrant. Then the value of 3α is equal to _______ .

माना प्रथम चतुर्थांश में वक्र $y^2 = 8x$ और रेखाओं $y = x$ एवं $x = 2$ से घिरे बड़े क्षेत्र का क्षेत्रफल α है। तो 3α का मान बराबर है।

Ans. Official Answer NTA (22)

Sol. On solving $y^2 = 8x$ and $y = x$, we get

$x = 0, 8$

Now shaded area = $\int_2^8 \left(2\sqrt{2}\cdot\sqrt{x} - x\right)dx$

$$= 2\sqrt{2}\int_2^8 \sqrt{x}\cdot dx - \int_2^8 x dx$$

$$= \frac{4\sqrt{2}}{3}\left[x^{\frac{3}{2}}\right]_2^8 - \frac{1}{2}\left[x^2\right]_2^8$$

$$=\frac{4\sqrt{2}}{3}\left[8^{\frac{3}{2}}-2^{\frac{3}{2}}\right]-\frac{1}{2}\left[8^{2}-2\right]$$

$$=\frac{4\sqrt{2}\times 2\sqrt{2}}{3}[8-1]-\frac{1}{2}\times 60$$

$$=\frac{8\times 2\times 7}{3}-30=\frac{22}{3}=\alpha$$

$$3\alpha=22$$

Question ID : 7155052076

27. Let $\sum_{n=0}^{\infty}\frac{n^{3}((2n)!)+(2n-1)(n!)}{(n!)((2n!))}=ae+\frac{b}{e}+c$, where $a,b,c\in Z$ and $e=\sum_{n=0}^{\infty}\frac{1}{n!}$. Then a^2-b+c is equal to

माना $\sum_{n=0}^{\infty}\frac{n^{3}((2n)!)+(2n-1)(n!)}{(n!)((2n!))}=ae+\frac{b}{e}+c$, है जहाँ $a,b,c\in Z$ तथा $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ है। तो a^2-b+c बराबर है।

Ans. Official Answer NTA (26)

Sol. $\sum_{n=0}^{\infty}\frac{n^{3}(2n)!}{n!(2n)!}+\sum_{n=0}^{\infty}\frac{(2n-1)n!}{n!(2n)!}$

$$\sum_{n=0}^{\infty}\frac{(2n-1)n!}{n!(2n)!}$$

$$\sum_{n=0}^{\infty}\frac{(2n)\times n!}{n!\times(2n)!}-\frac{n!}{n!(2n)!}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n-1)!}-\sum_{n=0}^{\infty}\frac{1}{(2n)!}$$

$$\left.\frac{e^{x}-e^{-x}}{2}\right|_{x=1}-\left.\frac{e^{x}+e^{-x}}{2}\right|_{x=1}$$

$$=\sum_{n=0}^{\infty}\frac{n^3(2n)!}{n!(2n)!}=\sum_{n=0}^{\infty}\frac{n^3}{n!}$$

$n^3 = an(n-1)(n-2) + bn(n-1) + cn + d$

$n = 1 \Rightarrow c = 1$

$n = 2 \Rightarrow 8 = 2b + 2 \Rightarrow b = 3$

$$=\sum_{n=0}^{\infty}\frac{n(n-1)(n-2)+3n(n-1)+n}{n!}$$

$$\sum_{n=3}^{\infty}\frac{1}{(n-3)!}+3\sum_{n=2}^{\infty}\frac{1}{(n-2)!}+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$$

$$=\sum_{n=0}^{\infty}\frac{n^3(2n)!}{n!(2n)!}+\sum_{n=0}^{\infty}\frac{(2n-1)n!}{n!(2n)!}$$

$$=5e+\frac{e}{2}-\frac{1}{e}-\frac{e}{2}+\frac{1}{\frac{e}{2}}$$

$$=5e-\frac{1}{e}$$

$a = 5, b = -1, c = 0$

Now $a^2 - b + c = 26$

Question ID : 7155052079

28. Let $f^1(x)=\frac{3x+2}{2x+3}, x\in R-\left\{\frac{-3}{2}\right\}$

For $n \ge 2$, define $f^n(x) = f^1 o f^{n-1}(x)$.

If $f^5(x)=\frac{ax+b}{bx+a}$, gcd(a,b) = 1, then a + b is equal to ________ .

माना $f^1(x)=\frac{3x+2}{2x+3}, x\in R-\left\{\frac{-3}{2}\right\}$ है।

$n \ge 2$, के लिए $f^n(x) = f^1 o f^{n-1}(x)$ द्वारा परिभाषित कीजिए। यदि $f^5(x)=\frac{ax+b}{bx+a}$, gcd(a,b) = 1 है, तो a + b बराबर है।

Ans. Official Answer NTA (3125)

Sol. $f^1(x) = \frac{3x+2}{2x+3}$

$\Rightarrow f^2(x) = \frac{13x+12}{12x+13}$

$\Rightarrow f^3(x) = \frac{63x+62}{62x+63}$

$\therefore f^5(x) = \frac{1563x+1562}{1562x+1563}$

a + b = 3125

Question ID : 7155052084

29. The mean and variance of 7 observations are 8 and 16 respectively. If one observation 14 is omitted and a and b are respectively mean and variance of remaining 6 observation, then a + 3b - 5 is equal to ______ .

7 प्रेक्षणों के माध्य तथा प्रसरण क्रमशः 8 तथा 16 है यदि एक प्रेक्षण 14 को हटाने पर शेष 6 प्रेक्षणों का माध्य तथा प्रसरण क्रमशः a तथा b हैं, तो a + 3b - 5 बराबर है।

Ans. Official Answer NTA (37)

Sol. Mean = 8, Variance = 16

Thus $\frac{\sum_{i=1}^{6} x_i + 14}{7} = 8$

$\sum_{i=1}^{6} x_i = 56 - 14 = 42$

Now, new mean, $a = \frac{\sum_{i=1}^{6} x_i}{6} = \frac{42}{6} = 7$

Also, $\frac{\sum_{i=1}^{6} x_i^2 + 14^2}{7} - 8^2 = 16$

$\sum_{i=1}^{6} x_i^2 = 560 - 196 = 364$

New variance = $b = \frac{\Sigma x_i^2}{6} - a^2$

$$= \frac{364}{6} - 7^2 = \frac{25}{3}$$

$$\therefore a + 3b - 5 = 7 + 3 \times \frac{25}{3} - 5$$

$$= 27$$

Question ID : 7155052077

30. Let $z = 1 + i$ and $z_1 = \dfrac{1 + i\bar{z}}{\bar{z}(1 - z) + \dfrac{1}{z}}$. Then $\dfrac{12}{\pi}\arg(z_1)$ is equal to ______ .

माना $z = 1 + i$ तथा $z_1 = \dfrac{1 + i\bar{z}}{\bar{z}(1 - z) + \dfrac{1}{z}}$. है तो $\dfrac{12}{\pi}\arg(z_1)$ बराबर है।

Ans. Official Answer NTA (9)

Sol. $z = 1 + i$

$$z_1 = \frac{1 + i\bar{z}}{\bar{z}(1 - z) + \frac{1}{z}}$$

$$z_1 = \frac{1 + i(1 - i)}{(1 - i)(1 - 1 - i) + \frac{1}{1 + I}}$$

$$= \frac{1 + i - i^2}{(1 - i)(-i) + \frac{1 - i}{2}}$$

$$= \frac{2 + i}{\frac{-3i - 1}{2}} = \frac{4 + 2i}{-3i - 1}$$

$$= \frac{-(4 + 2i)(3i - 1)}{(3i)^2 - (1)^2}$$

$$\operatorname{Arg}(z_1) = \frac{3\pi}{4}$$

$$\therefore \frac{12}{\pi}\arg(z_1) = \frac{12}{\pi} \times \frac{3\pi}{4} = 9$$