JEE Main January 2023

Question Paper With Text Solution

29 January | Shift-2

MATHEMATICS

JEE MAIN JANUARY 2023 | 29TH JANUARY SHIFT-2

SECTION – A

Question ID : 366694348

1. If $\vec{a}=\hat{i}+2\hat{k}$, $\vec{b}=\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=7\hat{i}-3\hat{j}+4\hat{k}$, $\vec{r}\times\vec{b}+\vec{b}\times\vec{c}=\vec{0}$ and $\vec{r}\cdot\vec{a}=0$. Then $\vec{r}\cdot\vec{c}$ is equal to

यदि $\vec{a}=\hat{i}+2\hat{k}$, $\vec{b}=\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=7\hat{i}-3\hat{j}+4\hat{k}$, $\vec{r}\times\vec{b}+\vec{b}\times\vec{c}=\vec{0}$ तथा $\vec{r}\cdot\vec{a}=0$ है। तो $\vec{r}\cdot\vec{c}$ बराबर है :

(1) 30 (2) 34 (3) 36 (4) 32

Ans. Official Answer NTA (2)

Sol. $\vec{r}\times\vec{b}-\vec{c}\times\vec{b}=0$

$\Rightarrow(\vec{r}-\vec{c})\times\vec{b}=0$

$\Rightarrow\vec{r}-\vec{c}=\lambda\vec{b}$

$\Rightarrow\vec{r}=\vec{c}+\lambda\vec{b}$

And given that $\vec{r}\cdot\vec{a}=0$

$\Rightarrow(\vec{c}+\lambda\vec{b})\cdot\vec{a}=0$

$\Rightarrow\vec{c}\cdot\vec{a}+\lambda\vec{b}\cdot\vec{a}=0$

$\Rightarrow\lambda=\dfrac{-\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}$

Now $\vec{r}\cdot\vec{c}=(\vec{c}++\lambda\vec{b})\cdot\vec{c}$

$=\left(\vec{c}-\dfrac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\vec{b}\right)\cdot\vec{c}$

$=|\vec{c}|-\left(\dfrac{\vec{c}\cdot\vec{a}}{\vec{b}\cdot\vec{a}}\right)(\vec{b}\cdot\vec{c})$

$=74-\left[\dfrac{15}{3}\right]8$

$=74-40=34$

Question ID : 366694336

2. Let f and g be twice differentiable functions on R such that

$f''(x) = g''(x) + 6x$

$f'(1) = 4g'(1) - 3 = 9$

$f(2) = 3g(2) = 12$

Then which of the following is NOT true?

(1) $g(-2) - f(-2) = 20$

(2) If $-1 < x < 2$, then $|f(x) - g(x)| < 8$

(3) There exists $x_0 \in (1, 3/2)$ such that $f(x_0) = g(x_0)$

(4) $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$

माना फलन f तथा g, R पर दो बार अवकलनीय हैं तथा

$f''(x) = g''(x) + 6x$

$f'(1) = 4g'(1) - 3 = 9$

$f(2) = 3g(2) = 12$

हैं। तो निम्न में से कौन सा सत्य नहीं है ?

(1) $g(-2) - f(-2) = 20$ है

(2) यदि $-1 < x < 2$ है, तो $|f(x) - g(x)| < 8$ है

(3) $x_0 \in (1, 3/2)$ का अस्तित्व है, जिसके लिए $f(x_0) = g(x_0)$ है

(4) $|f'(x) - g'(x)| < 6 \Rightarrow -1 < x < 1$

Ans. Official Answer NTA (2)

Sol. $f''(x) = g''(x) + 6x$

$\Rightarrow \int f''(x)dx = \int g''(x)dx + \int 6x dx + C$

$\Rightarrow f'(x) = g'(x) + 3x^2 + C$ ....(1)

$x = 1 \Rightarrow f'(1) = g'(1) + 3 + C$

$\Rightarrow 9 = 3 + 3 + C \Rightarrow C = 3$

from (1) $\int f'(x)dx = \int g'(x)dx + \int 3x^2 dx + \int 3dx + k$

$\Rightarrow f(x) = g(x) + x^3 + 3x + K$

$x = 2 \Rightarrow f(2) = g(2) + 8 + 6 + K$

$\Rightarrow 12 = 4 + 8 + 6 + K$

$\Rightarrow K = -6$

$\Rightarrow f(x) = g(x) + x^3 + 3x - 6$

Let $h(x) = f(x) - g(x) = x^3 + 3x - 6$

$\Rightarrow h'(x) = f'(x) - g'(x) = 3x^2 + 3 > 0 \ \forall \ x$

$\Rightarrow h'(x)$ is S.I. $\Rightarrow h(x) = 0$ has no solution

$\Rightarrow h(x) \in (h(-1), h(2)) \Rightarrow h(x) \in (-8, 8)$

$|f'(x) - g'(x)| < 6 \Rightarrow |3x^2 + 3| < 6 \Rightarrow 3x^2 + 3 < 6$

$\Rightarrow x^2 - 1 < 0 \Rightarrow x \in (-1, 1)$

$h(-2) = f(-2) - g(-2) = -20$

Question ID : 366694346

3. Let $\vec{a} = 4\hat{i} + 3\hat{j}$ and $\vec{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$. If $\vec{c}$ is a vector such that $\vec{c}\cdot\left(\vec{a}\times\vec{b}\right) + 25 = 0$, $\vec{c}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 4$, and projection of $\vec{c}$ on $\vec{a}$ is 1, then the projection of $\vec{c}$ on $\vec{b}$ equals

माना $\vec{a} = 4\hat{i} + 3\hat{j}$ तथा $\vec{b} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ है। यदि एक सदिश $\vec{c}$ के लिए $\vec{c}\cdot\left(\vec{a}\times\vec{b}\right) + 25 = 0$, $\vec{c}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right) = 4$ हैं तथा $\vec{c}$ का $\vec{a}$ पर प्रक्षेप 1 है, तो $\vec{c}$ का $\vec{b}$ पर प्रक्षेप बराबर है :

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{5}$ (3) $\frac{3}{\sqrt{2}}$ (4) $\frac{5}{\sqrt{2}}$

Ans. Official Answer NTA (4)

Sol. $\vec{a}\times\vec{b} = 15\hat{i} - 20\hat{j} - 25\hat{k}$

Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow 15x - 20y - 25z + 25 = 0$

$\Rightarrow 3x - 4y - 5z = -5$

Also $x + y + z = 4$

and $\frac{\vec{c}\cdot\vec{a}}{|\vec{a}|} = 1 \Rightarrow 4x + 3y = 5$

$\Rightarrow \vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}$

Projection of $\vec{c}$ or $\vec{b} = \frac{25}{5\sqrt{2}} = \frac{5}{\sqrt{2}}$

Question ID : 366694341

4. The number of 3 digit numbers, that are divisible by either 3 or 4 but not divisible by 48, is

तीन अंकों की संख्याओं, जो 3 या 4 से विभाज्य है परन्तु 48 से विभाज्य नहीं है, की संख्या है

(1) 432 (2) 400 (3) 472 (4) 507

Ans. Official Answer NTA (1)

Sol. no. divisible by 3 = 102, 105, ... , 999

= 300

no. divisible by 4 = 100, 104, ... , 996

= 225

now no. divisible by 12

= 75

So numbers divisible by 3 or 4

= 300 + 225 – 75 = 450

now numbers divisible by 48 are = 18

So total numbers = 450 – 18

= 432

Question ID : 366694342

5. If the tangent at a point P on the parabola $y^2 = 3x$ is parallel to the line $x + 2y = 1$ and the tangents at the points Q and R on the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$ are perpendicular to the line $x - y = 2$, then the area of the triangle PQR is :

यदि परवलय $y^2 = 3x$ के बिंदु P पर स्पर्श रेखा, सरल रेखा $x + 2y = 1$ के समांतर है तथा दीर्घवृत्त $\frac{x^2}{4} + \frac{y^2}{1} = 1$ के बिंदुओं Q तथा R पर स्पर्श रेखाएँ, सरल रेखा $x - y = 2$ के लंबवत है, तो त्रिभुज PQR का क्षेत्रफल है :

(1) $3\sqrt{5}$ (2) $5\sqrt{3}$ (3) $\frac{3}{2}\sqrt{5}$ (4) $\frac{9}{\sqrt{5}}$

Ans. Official Answer NTA (1)

Sol. $y^2 = 3x$

Tangent $P(x_1, y_1)$ is parallel to $x + 2y = 1$

Then slope at P $= -\frac{1}{2}$

$\Rightarrow 2y\frac{dy}{dx} = 3$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2y} = -\frac{1}{2}$

$\Rightarrow y_1 = -3$

Coordinates of P(3, –3)

Similarly $Q\left(\frac{4}{\sqrt{3}}, \frac{1}{\sqrt{5}}\right)$, $R\left(-\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$

Area of ΔPQR

$$= \frac{1}{2}\begin{vmatrix} 3 & -3 & 1 \\ \frac{4}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 \\ -\frac{4}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[3\left(\frac{2}{\sqrt{5}}\right) + 3\left(\frac{8}{\sqrt{5}}\right) + 0\right] = \frac{30}{2\sqrt{5}} = 3\sqrt{5}$$

Question ID : 366694332

6. The set of all values of $t \in R$ for which the matrix

$$\begin{bmatrix} e^t & e^{-1}(\text{sint-2cost}) & e^{-1}(\text{-2sint-cost}) \\ e^t & e^{-1}(\text{2sint+cost}) & e^{-t}(\text{sint-2cost}) \\ e^t & e^{-1}\text{cost} & e^{-t}\text{sint} \end{bmatrix}$$ is invertible, is

$t \in R$ के सभी मानों, जिनके लिए आव्यूह

$$\begin{bmatrix} e^t & e^{-1}(\text{sint-2cost}) & e^{-1}(\text{-2sint-cost}) \\ e^t & e^{-1}(\text{2sint+cost}) & e^{-t}(\text{sint-2cost}) \\ e^t & e^{-1}\text{cost} & e^{-t}\text{sint} \end{bmatrix}$$ व्युत्क्रमणीय है, का समुच्चय है :

(1) R (2) $\{k\pi, k \in Z\}$ (3) $\left\{(2k+1)\frac{\pi}{2}, k \in Z\right\}$ (4) $\left\{k\pi + \frac{\pi}{4}, k \in Z\right\}$

Ans. Official Answer NTA (1)

Sol. $$|A| = \begin{bmatrix} e^t & e^{-t}(\sin t - 2\cos t) & e^{-t}(-2\sin t - \cos t) \\ e^t & e^{-t}(2\sin t + \cos t) & e^{-t}(\sin t - 2\cos t) \\ e^t & e^{-t}\cos t & e^{-t}\sin t \end{bmatrix}$$

$$= e^{t}.e^{-t}.e^{-t}\begin{vmatrix} 1 & \sin t - 2\cos t & -2\sin t - \cos t \\ 1 & 2\sin t + \cos t & \sin t - 2\cos t \\ 1 & \cos t & \sin t \end{vmatrix}$$

$R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$|A| = e^{-t}\begin{vmatrix} 0 & \sin t - 3\cos t & -3\sin t - \cos t \\ 0 & 2\sin t & -2\cos t \\ 1 & \cos t & \sin t \end{vmatrix}$$

$= e^{-t}(-2\cos t(\sin t - 3\cos t) + 2\sin t(3\sin t + \cos t))$
$= e^{-t}(6\cos^2 t + 6\sin^2 t) = 6e^{-t}$
$|A| \neq 0 \ \forall \ t \in R$

Question ID : 366694349

7. The set of all values of λ for which the equation $\cos^2 2x - 2\sin^4 x - 2\cos^2 x = \lambda$ has a real solution x, is

λ के सभी मानों, जिनके लिए समीकरण $\cos^2 2x - 2\sin^4 x - 2\cos^2 x = \lambda$ का एक वास्तविक हल x है, का समुच्चय है :

(1) $\left[-1, -\frac{1}{2}\right]$ (2) $\left[-2, -\frac{3}{2}\right]$ (3) [-2,-1] (4) $\left[-\frac{3}{2}, -1\right]$

Ans. Official Answer NTA (4)

Sol. $\lambda = \cos^2 2x - 2\sin^4 x - 2\cos^2 x$

convert all in to cos x.

$\lambda = (2\cos^2 x - 1)^2 - 2(1 - \cos^2 x)^2 - 2\cos^2 x$
$= 4\cos^4 x - 4\cos^2 x + 1 - 2(1 - 2\cos^2 x + \cos^4 x) - 2\cos^2 x$
$= 2\cos^4 x - 2\cos^2 x + 1 - 2$
$= 2\cos^4 x - 2\cos^2 x - 1$

$$= 2\left[\cos^4 x - \cos^2 x - \frac{1}{2}\right]$$

$$= 2\left[\left(\cos^2 x - \frac{1}{2}\right)^2 - \frac{3}{4}\right]$$

$$\lambda_{max} = 2\left[\frac{1}{4} - \frac{3}{4}\right] = 2\times\left(-\frac{2}{4}\right) = -1 \text{ (max Value)}$$

$$\lambda_{min} = 2\left[0 - \frac{3}{4}\right] = -\frac{3}{2} \text{ (minimum value)}$$

So, Range $=\left[-\frac{3}{2},-1\right]$

Question ID : 366694334

8. Let K be the sum of the coefficients of the odd powers of *x* in the expansion of $(1+x)^{99}$. Let *a* be the middle term in the expansion of $\left(2+\frac{1}{\sqrt{2}}\right)^{200}$. If $\frac{{}^{200}C_{99}K}{a}=\frac{2^{\ell}m}{n}$, where *m* and *n* are odd numbers, then the ordered pair (*l*, n) is equal to

माना $(1+x)^{99}$ के प्रसार में x की विषम घातों के गुणांको का योग K है। माना $\left(2+\frac{1}{\sqrt{2}}\right)^{200}$ के प्रसार में मध्य पद a है। यदि $\frac{{}^{200}C_{99}K}{a}=\frac{2^{\ell}m}{n}$ है, जहाँ *m* तथा *n* विषम संख्याऐं है, तो क्रमित युग्म (*l*, n) बराबर है।

(1) (50, 51) (2) (51,101) (3) (50,101) (4) (51,99)

Ans. Official Answer NTA (3)

Sol. Sum of odd coeff. of $x=\frac{2^{99}-0}{2}=k$

$k = 2^{98}$

Middle term $\alpha = T101 = {}^{200}C_{100}\frac{(2)^{100}}{2^{50}}$

$= {}^{200}C_{100}\ 2^{50}$

Now $\frac{{}^{200}C^{99}\times k}{\alpha}=\frac{200!}{100!99!}\times\frac{2^{98}}{{}^{200}C_{100}\times 2^{50}}$

$=\frac{100}{101}\times 2^{48}$

$\frac{25}{101}\times 2^{50}$

Question ID : 366694338

9. The value of the integral $\int_{1/2}^{2}\frac{\tan^{-1}x}{x}dx$ is equal to

समाकलन $\int_{1/2}^{2}\frac{\tan^{-1}x}{x}dx$ का मान बराबर है

(1) $\pi\log_e 2$ (2) $\frac{1}{2}\log_e 2$ (3) $\frac{\pi}{4}\log_e 2$ (4) $\frac{\pi}{2}\log_e 2$

Ans. Official Answer NTA (4)

Sol. $I=\int_{1/2}^{2}\frac{\tan^{-1}x}{x}dx$ ......(i)

Put $x=\frac{1}{t}$ $dx=-\frac{1}{t^2}dt$

$$I=-\int_{2}^{1/2}\frac{\tan^{-1}\frac{1}{t}}{\frac{1}{t}}\cdot\frac{1}{t^2}dt=-\int_{2}^{1/2}\frac{\tan^{-1}\frac{1}{t}}{t}dt$$

$$I=\int_{2}^{1/2}\frac{\cot^{-1}t}{t}dt=\int_{2}^{1/2}\frac{\cot^{-1}x}{x}dx$$ ....(ii)

Add both equation

$$2I=\int_{1/2}^{2}\frac{\tan^{-1}x+\cot^{-1}x}{x}dx=\frac{\pi}{2}\int_{1/2}^{2}\frac{dx}{x}=\frac{\pi}{2}(\ell n\,2)_{1/2}^{2}$$

$$=\frac{\pi}{2}\left(\ell n2-\ell n\frac{1}{2}\right)=\pi\ell n2$$

$$I=\frac{\pi}{2}\ell n2$$

Question ID : 366694350

10. The statement $B\Rightarrow((\sim A)\vee B)$ is equivalent to :

कथन $B\Rightarrow((\sim A)\vee B)$ निम्न में से किस के तुल्य है :

(1) $B\Rightarrow(A\Rightarrow B)$ (2) $A\Rightarrow(A\Leftrightarrow B)$ (3) $B\Rightarrow((\sim A)\Rightarrow B)$ (4) $A\Rightarrow((\sim A)\Rightarrow B)$

Ans. Official Answer NTA (1,3,4)

Sol. $B\rightarrow(\sim A\vee B)=B\rightarrow(A\rightarrow B)=\sim B\vee(\sim A\vee B)=t$

Similarly $B \Rightarrow ((\sim A) \Rightarrow B) = t$ and $A \Rightarrow ((\sim A) \Rightarrow B) = t$

Question ID : 366694344

11. If the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{1}$ and $\frac{x-a}{2}=\frac{y+2}{3}=\frac{z-3}{1}$ intersect at the point P, then the distance of the point P from the plane $z=a$ is :

यदि रेखाएें $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z+3}{1}$ तथा $\frac{x-a}{2}=\frac{y+2}{3}=\frac{z-3}{1}$ बिंदु P पर मिलती हैं, तो बिंदु P की समतल $z=a$ से दूरी है :

(1) 28 (2) 16 (3) 10 (4) 22

Ans. Official Answer NTA (1)

Sol. Point on $L_1 \equiv (\lambda + 1, 2\lambda + 2, \lambda - 3)$

Point on $L_2 \equiv (2\mu + a, 3\mu - 2, \mu + 3)$

$\lambda - 3 = \mu + 3 \quad \Rightarrow \lambda = \mu + 6 \quad ...(1)$

$2\lambda + 2 = 3\mu - 2 \quad \Rightarrow 2\lambda = 3\mu - 4 \quad ...(2)$

Solving, (1) and (2)

$\Rightarrow \lambda = 22$ & $\mu = 16$

$\Rightarrow P \equiv (23, 46, 19)$

$\Rightarrow a = -9$

Distance of P from $z = -9$ is 28

Question ID : 366694331

12. Let R be a relation defined on N as a R b if $2a + 3b$ is a multiple of 5, $a, b \in N$. Then R is

(1) not reflexive

(2) an equivalence relation

(3) symmetric but not transitive

(4) transitive but not symmetric

N पर एक संबंध R, a R b यदि $2a + 3b$ का एक गुणज है द्वारा परिभाषित है, तो R

(1) स्वतुल्य नहीं है

(2) एक तुल्यता संबंध है

(3) संक्रामक है परन्तु सममित नहीं है

(4) सममित है परन्तु संक्रामक नहीं है

Ans. Official Answer NTA (2)

Sol. For (A, a) $\Rightarrow 2a + 3b = 2a + 3b = 5a \Rightarrow$ divisible by 5

$\Rightarrow (a, a) \in R\ \forall\ a \in N \Rightarrow$ reflexive

Let $(a, b) \in R \Rightarrow 2a + 3b = 5k_1$

and $5a + 5b = 5k_2$ then

$5a + 5b - 2a - 3b = 5(k_2 - k_1) \Rightarrow 2b + 3a = 5k$

$\Rightarrow (b, a) \in R \Rightarrow$ symmetric

Let (a, b) & (b, c) both belong to R

$\Rightarrow 2a + 3b = 5k_1$ & $2b + 3c = 5k_2$

then $2a + 3b + 2b + 3c = 5(k_1 + k_2) \Rightarrow 2a + 3c = 5k - 5b = 5\lambda$

$\Rightarrow (a, c) \in R \Rightarrow$ transitive

Question ID : 366694345

13. The plane $2x - y + z = 4$ intersects the line segment joining the points A $(a, -2, 4)$ and B $(2, b, -3)$ at the point C in the ratio 2 : 1 and the distance of the point C from the origin is $\sqrt{5}$. If $ab < 0$ and P is the point $(a - b, b, 2b - a)$ then CP^2 is equal to

समतल $2x - y + z = 4$ बिंदुओं A $(a, -2, 4)$ तथा B $(2, b, -3)$ को मिलाने वाले रेखाखंड को 2 : 1 के अनुपात में बिंदु C पर काटता है तथा मूलबिंदु से बिंदु C की दूरी $\sqrt{5}$ है। यदि $ab < 0$ है तथा P, बिंदु $(a - b, b, 2b - a)$ है, तो CP^2 बराबर है :

(1) $\frac{16}{3}$ (2) $\frac{97}{3}$ (3) $\frac{17}{3}$ (4) $\frac{73}{3}$

Ans. Official Answer NTA (3)

Sol. A(a, –2, 4), B(2, b, –3)

AC : CB = 2 : 1

$$\Rightarrow C \equiv \left(\frac{a+4}{3}, \frac{2b-2}{3}, \frac{-2}{3}\right)$$

C lies on $2x - y + 2 = 4$

$$\Rightarrow \frac{2a+8}{3} - \frac{2b-2}{3} - \frac{2}{3} = 4$$

$\Rightarrow a - b = 2$ ....(1)

Also OC = $\sqrt{5}$

$$\Rightarrow \left(\frac{a+4}{3}\right)^2 + \left(\frac{2b-2}{3}\right)^2 + \frac{4}{9} = 5 \qquad ....(2)$$

Solving, (1) and (2)

$(b + 6)^2 + (2b - 2)^2 = 41$

$\Rightarrow 5b^2 + 4b - 1 = 0$

$\Rightarrow b = -1 \text{ or } \frac{1}{5}$

$\Rightarrow a = 1 \text{ or } \frac{11}{5}$

But $ab < 0 \Rightarrow (a, b) = (1, -1)$

$C \equiv \left(\frac{5}{3}, \frac{-4}{3}, \frac{-2}{3}\right),\ P \equiv (2, -1, -3)$

$CP^2 = \frac{1}{9} + \frac{1}{9} + \frac{49}{9} = \frac{51}{9} = \frac{17}{3}$

Question ID : 366694335

14. The letters of the word OUGHT are written in all possible ways and these words are arranged in a dictionary, in a series. Then the serial number of the word TOUGH is

शब्द OUGHT के अक्षरों को सभी संभव तरीकों में लिखा जाता है तथा इन शब्दों को एक शब्दकोश की तरह एक श्रेणी में व्यवस्थित किया जाता है। तो शब्द TOUGH की एक क्रम संख्या है :

(1) 89 (2) 79 (3) 86 (4) 84

Ans. Official Answer NTA (1)

Sol. G, H, O, T, U

No. of words starting from G = 4! = 24

No. of words starting from H = 4! = 24

No. of words starting from O = 4! = 24

No. of words starting from TG = 3! = 6

No. of words starting from TH = 3! = 6

No. of words starting from TOG = 2! = 2

Next "TOUGH" = 1

Rank of "TOUGH" = 24 + 24 + 24 + 6 + 6 + 2 + 2 + 1 = 89

Question ID : 366694340

15. Let $y=y(x)$ be the solution of the differential equation $x\log_e x\dfrac{dy}{dx}+y=x^2\log_e x$, $(x>1)$. If $y(2)=2$, then $y(e)$ is equal to

माना अवकल समीकरण $x\log_e x\dfrac{dy}{dx}+y=x^2\log_e x$, $(x>1)$ का हल $y=y(x)$ है। यदि $y(2)=2$ है, तो $y(e)$ बराबर है :

(1) $\dfrac{1+e^2}{2}$ (2) $\dfrac{1+e^2}{4}$ (3) $\dfrac{4+e^2}{4}$ (4) $\dfrac{2+e^2}{2}$

Ans. Official Answer NTA (3)

Sol. $x\log_e x\dfrac{dy}{dx}+y=x^2\log_e x, (x>1)$

$\Rightarrow \dfrac{dy}{dx}+\dfrac{y}{x\ln x}=x$

Linear differential equation

$$\text{I.F.} = e^{\int\frac{1}{x\ln x}dx}=|\ln x|$$

$\therefore$ Solution of differential equation

$$y|\ln x|=\int x|\ln x|dx$$

$$|\ln x^2|\frac{x^2}{2}-\int\frac{1}{x}\cdot\frac{x^2}{2}dx$$

$$\Rightarrow y|\ln x|=|\ln x|\left(\frac{x^2}{2}\right)-\frac{x^2}{4}+c$$

For constant

$y(2)=2\Rightarrow c=1$

So, $y(x)=\dfrac{x^2}{2}-\dfrac{x^2}{4|\ln x|}+\dfrac{1}{|\ln x|}$

Hence, $y(e)=\dfrac{e^2}{2}-\dfrac{e^2}{4}+1=1+\dfrac{e^2}{4}$

Question ID : 366694339

16. The area of the region $A=\left\{(x,y):|\cos x-\sin x|\le y\le \sin x, 0\le x\le\frac{\pi}{2}\right\}$ is

क्षेत्र $A = \left\{(x, y) : |\cos x - \sin x| \le y \le \sin x, 0 \le x \le \frac{\pi}{2}\right\}$ का क्षेत्रफल है :

(1) $\sqrt{5} - 2\sqrt{2} + 1$

(2) $1 - \frac{3}{\sqrt{2}} + \frac{4}{\sqrt{5}}$

(3) $\frac{3}{\sqrt{5}} - \frac{3}{\sqrt{2}} + 1$

(4) $\sqrt{5} + 2\sqrt{2} - 4.5$

Ans. Official Answer NTA (1)

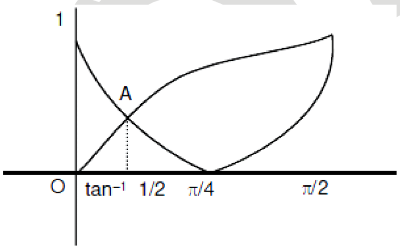
Sol.
$$y = \begin{cases} \cos x - \sin x & : \quad 0 \le x \le \frac{\pi}{4} \\ \sin x - \cos x & : \quad \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$$

$$y^1 = \begin{cases} -\sin x - \cos x & : \quad 0 \le x \le \frac{\pi}{4} \\ \cos x + \sin x & : \quad \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$$

For pt. A

$\cos x - \sin x = \sin x$

$\tan x = \frac{1}{2}$



Let $x = \tan^{-1}\frac{1}{2}$

$$\text{Area} = \int_{\tan^{-1}1/2}^{\pi/4} (\sin x - (\cos x - \sin x))dx + \int_{\pi/4}^{\pi/2} (\sin x - \sin x + \cos x)dx$$

$$= (-2\cos x - \sin x)_x^{\pi/4} + (\sin x)_{\pi/4}^{\pi/2}$$

$$= \left(\frac{-2}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - (-2\cos x - \sin x) + \left(1 - \frac{1}{\sqrt{2}}\right)$$

$= -\sqrt{2} - \frac{1}{\sqrt{2}} + 2\cos x + \sin x + 1 - \frac{1}{\sqrt{2}} = -2\sqrt{2} + 1 + \frac{2\times 2}{\sqrt{5}} + \frac{1}{\sqrt{5}}$

$= \sqrt{5} + 1 - 2\sqrt{2}$

Question ID : 366694343

17. The shortest distance between the lines $\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5}$ and $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3}$ is

रेखाओं $\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5}$ तथा $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3}$ के बीच न्यूनतम दूरी है :

(1) $4\sqrt{3}$ (2) $3\sqrt{3}$ (3) $5\sqrt{3}$ (4) $2\sqrt{3}$

Ans. Official Answer NTA (1)

Sol. $\frac{x-1}{2} = \frac{y+8}{-7} = \frac{z-4}{5}$ $\vec{a} = \hat{i} - 8\hat{j} + 4\hat{k}$

$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-6}{-3}$ $\vec{b} = \hat{i} + 2\hat{j} + 6\hat{k}$

$\vec{p} = 2\hat{i} - 7\hat{i} + 5\vec{\hat{k}}, \vec{q} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\vec{p}\times\vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix}$

$= \hat{i}(16) - \hat{j}(-16) + \hat{k}(16)$

$= 16(\hat{i} + \hat{j} + \hat{k})$

$d = \left|\frac{(a-b)\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right| = \left|\frac{(-10\hat{j} - 2\hat{k})\cdot 16(\hat{i} + \hat{j} + \hat{k})}{16\sqrt{3}}\right|$

$= \left|\frac{-12}{\sqrt{3}}\right| = 4\sqrt{3}$

Question ID : 366694333

18. Consider a function f: IN $\to$ IR , satisfying

$f(1) + 2f(2) + 3f(3) + ... + xf(x) = x(x+1)f(x);\ x \geq 2$ with $f(1) = 1$.

Then $\frac{1}{f(2022)}+\frac{1}{f(2028)}$ is equal to

एक फलन $f\colon \mathbb{N} \to \mathbb{R}$, के लिए $f(1)+2f(2)+3f(3)+\ldots+xf(x)=x(x+1)f(x);\ x \geq 2$ तथा $f(1)=1$ है।

तो $\frac{1}{f(2022)}+\frac{1}{f(2028)}$ बराबर है :

(1) 8000 (2) 8400 (3) 8100 (4) 8200

Ans. Official Answer NTA (3)

Sol. $f(1)+2f(2)+3f(3)\ldots\ldots\ldots\ldots\ n f(n)=n(n+1)f(n)$ ....(1)

replace in by n + 1

$f(1)+2f(2)+3f(3)\ldots\ldots\ldots\ nf(n)+(n+1)f(n+1)=(n+1)(n+2)f(n+1)$ ....(2)

Subtracting (1) from (2)

$(n+1)f(n+1)=(n+1)(n+2)f(n+1)-n(n+1)f(n)$ or $(n+1)f(n+1)-nf(n)=0$ ....(3)

put n = 2, 3, 4 .... n + 1 in eq. (3)

$3f(3)-2f(2)=0$

$4f(4)-3f(3)=0$

,,

$(n+1)f(n+1)-nf(n)=0$

Add $\Rightarrow (n+1)f(n+1)=2f(2)$

$\Rightarrow \quad f(n+1)=\frac{2f(2)}{n+1}$ ....(4)

again put n = 2 in eq. (1)

$f(1)+2f(2)=6f(2)$

$\Rightarrow 1=4f(2) \Rightarrow f(2)=\frac{1}{4}$

Now by (4) $f(n)=\frac{1}{2n}$

$\Rightarrow \frac{1}{f(2022)}+\frac{1}{f(2028)}=4044+4056=8100$

Question ID : 366694347

19. Let S = $\{w_1, w_2, \ldots\ldots\}$ be the sample space associated to a random experiment. Let $P(w_n)=\frac{P(w_{n-1})}{2}$,

$n \geq 2$. Let $A = \{2k + 3l : k, \ell \in N\}$ and $B = \{w_n : n \in A\}$. Then P(B) is equal to

माना एक यादृच्छिक परीक्षण की प्रतिदर्श समष्टि $S = \{w_1, w_2, ...\}$ है। माना $P(w_n) = \frac{P(w_{n-1})}{2}$, $n \geq 2$ है। माना $A = \{2k + 3l : k, \ell \in N\}$ तथा $B = \{w_n : n \in A\}$ है। तो P(B) बराबर है :

(1) $\frac{3}{32}$ (2) $\frac{1}{16}$ (3) $\frac{1}{32}$ (4) $\frac{3}{64}$

Ans. Official Answer NTA (4)

Sol. Let $P(w_1) = \lambda$ then $P(w_2) = \frac{\lambda}{2} ... P(w_n) = \frac{\lambda}{2^{n-1}}$

As $\sum_{k=1}^{\infty} P(w_k) = 1 \Rightarrow \frac{\lambda}{1 - \frac{1}{2}} = 1 \Rightarrow \lambda = \frac{1}{2}$

So, $P(w_n) = \frac{1}{2^n}$

$A = \{2k + 3\ell; k, \ell \in \mathbb{N}\} = \{5, 7, 8, 9, 10.....\}$

$B = \{w_n : n \in A\}$

$B = \{w_5, w_7, w_8, w_9, w_{10}, w_{11}, ...\}$

$A = \mathbb{N} - \{1, 2, 3, 4, 6\}$

$\therefore P(B) = 1 - [P(w_1) + P(w_2) + P(w_3) + P(w_4) + P(w_6)]$

$= 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{64}\right]$

$= 1 - \frac{32 + 16 + 8 + 4 + 1}{64} = \frac{3}{64}$

Question ID : 366694337

20. The value of the integral $\int_1^2 \left(\frac{t^4 + 1}{t^6 + 1}\right) dt$ is

समाकलन $\int_1^2 \left(\frac{t^4 + 1}{t^6 + 1}\right) dt$ का मान है :

(1) $\tan^{-1}\frac{1}{2} - \frac{1}{3}\tan^{-1} 8 + \frac{\pi}{3}$ (2) $\tan^{-1}\frac{1}{2} + \frac{1}{3}\tan^{-1} 8 - \frac{\pi}{3}$

(3) $\tan^{-1}2-\frac{1}{3}\tan^{-1}8+\frac{\pi}{3}$ (4) $\tan^{-1}2+\frac{1}{3}\tan^{-1}8-\frac{\pi}{3}$

Ans. Official Answer NTA (4)

Sol. $=\int_1^2\frac{(t^4+1)}{(t^6+1)}\frac{(t^2+1)}{(t^2+1)}dt$

$=\int_1^2\frac{t^6+t^4}{(t^6+1)}\frac{t^2+1}{(t^2+1)}dt$

$=\int_1^2\frac{t^6+1}{(t^6+1)(t^2+1)}dt+\int_1^2\frac{t^4+t^2}{(t^6+1)(t^2+1)}dt$

$=\int_1^2\frac{1}{t^2+1}dt+\int_1^2\frac{t^2}{t^6+1}dt,$

$I=\left[\tan^{-1}t\right]_1^2+I_1$

$\tan^{-1}-\frac{\pi}{4}+I_1\ \tan^{-1}-\frac{\pi}{4}+I_1$

$I_1=\int_{11}^2\frac{t^2}{t^6+1}dt$

put $t^3=\theta\cdot 3t^2\,dt=d\theta$
$t=1$ then $\theta=1$
$t=2$ then $\theta=8$

$=\frac{1}{3}\int_1^8\frac{d\theta}{\theta^2+1}$

$=\frac{1}{3}\left[\tan^{-1}(\theta)\right]_1^8$

$=\frac{1}{3}\left[\tan^{-1}8-\frac{\pi}{4}\right]$

$I=\tan^{-1}2-\tan^{-1}(1)+\frac{1}{3}\tan^{-1}8-\frac{\pi}{12}$

$\tan^{-1}2+\frac{1}{3}\tan^{-1}8-\frac{\pi}{3}$

SECTION – B

Question ID : 366694359

21. A triangle is formed by the tangents at the point (2,2) on the curves $y^2 = 2x$ and $x^2 + y^2 = 4x$, and the line $x + y + 2 = 0$. If r is the radius of its circumcircle, then r^2 is equal to ______ .

वक्रों $y^2 = 2x$ तथा $x^2 + y^2 = 4x$ के बिंदु (2, 2) पर स्पर्श रेखाएँ तथा रेखा $x + y + 2 = 0$ एक त्रिभुज बनाती है। यदि इस त्रिभुज के परिवृत्त की त्रिज्या r है, तो r^2 बराबर है।

Ans. Official Answer NTA (10)

Sol. $S_1 : y^2 = 2x$ $\quad$ $S2 : x^2 + y^2 = 4x$

P(2,2) is common point on S_1 & S_2

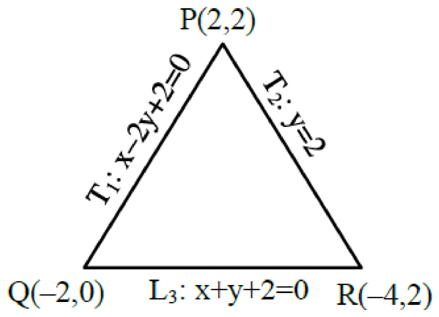
T_1 is tangent to S_1 at P $\Rightarrow T_1 : y.2 = x + 2$

$\Rightarrow T_1 : x - 2y + 2 = 0$

T_2 is tangent to S_2 at P $\Rightarrow T_2 : x.2 + y.2 = 2\ (x + 2)$

$\Rightarrow T_2 : y = 2$

& $L_3 : x + y + 2 = 0$ is third line



$PQ = a = \sqrt{20}$

$QR = b = \sqrt{8}$

$RP = c = 6$

Area (ΔPQR) = $\Delta = \frac{1}{2}\times 6\times 2 = 6$

$\therefore r = \frac{abc}{4\Delta} = \frac{\sqrt{160}}{4} = \sqrt{10} \Rightarrow r^2 = 10$

Question ID : 366694351

22. Let $\alpha_1, \alpha_2, ..., \alpha_7$ be the roots of the equation $x^7 + 3x^5 - 13x^3 - 15x = 0$ and $|\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_7|$. Then $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ is equal to ___________ .

माना समीकरण $x^7 + 3x^5 - 13x^3 - 15x = 0$ के मूल $\alpha_1, \alpha_2, ..., \alpha_7$ हैं तथा $|\alpha_1| \geq |\alpha_2| \geq ... \geq |\alpha_7|$. है। तो $\alpha_1\alpha_2 - \alpha_3\alpha_4 + \alpha_5\alpha_6$ बराबर है।

Ans. Official Answer NTA (9)

Sol. $x^7 + 3x^5 - 13x^3 - 15x = 0$

$\Rightarrow \quad x = 0, x^6 + 3x^4 - 13x^2 - 15 = 0$

Let $x^2 = t$

Equation $\quad t^3 + 3t^2 - 13t - 15 = 0$

$\Rightarrow (t - 3)(t^2 + 6t + 5) = 0$

$\Rightarrow t = 3, t^2 + 6t + 5 = 0$

$\Rightarrow t = -1, -5$

$x^2 = 3 \Rightarrow x = \sqrt{3}, -\sqrt{3}$

$x^2 = -1 \Rightarrow x = i, -i$

$x^2 = -5 \Rightarrow x = -\sqrt{5}i, \sqrt{5}i$

$a_1a_2 - a_3a_4 + a_5a_6$

$= \left(\sqrt{5}i\right)\left(-\sqrt{5}i\right) - \left(\sqrt{3}\right)\left(-\sqrt{3}\right) + (i)(-i)$

$= 5 + 3 + 1 = 9$

Question ID : 366694360

23. Let $X = \{11,12,13,.....,40,41\}$ and $Y = \{61,62,63,...,90,91\}$ be the two sets of observations. If $\bar{x}$ and $\bar{y}$ are their respective means and σ^2 is the variance of all the observations in $X \cup Y$, then $|\bar{x} + \bar{y} - \sigma^2|$ is equal to _______ .

माना प्रेक्षणों के दो समुच्चय $X = \{11,12,13,.....,40,41\}$ तथा $Y = \{61,62,63,...,90,91\}$ है। यदि इनके माध्य क्रमशः $\bar{x}$ तथा $\bar{y}$ हैं तथा $X \cup Y$ में सभी प्रेक्षणों का प्रसरण σ^2 है, तो $|\bar{x} + \bar{y} - \sigma^2|$ बराबर है।

Ans. Official Answer NTA (603)

Sol. $\bar{x} = \dfrac{\sum_{i=11}^{41} i}{31} = \dfrac{11+41}{2} = 26$ (31 elements)

$$\bar{y} = \frac{\sum_{j=61}^{91} j}{31} = \frac{61+91}{2} = 76 \text{ (31 elements)}$$

Combined mean, $\mu = \frac{31\times 26 + 31\times 76}{31+31}$

$= \frac{26+76}{2} = 51$

$$\sigma^2 = \frac{1}{62}\times\left(\sum_{i=1}^{31}(x_i-\mu)^2 + \sum_{i=1}^{31}(y_i-\mu)^2\right) = 705$$

Since, $x_i \in X$ are in A.P. with 31 elements & common difference 1, same is $y_i \in y$, when written in increasing order.

$\therefore \sum_{i=1}^{31}(x_i-\mu) = \sum_{i=1}^{31}(y_i-\mu)^2$

$= 10^2 + 11^2 + ... + 40^2$

$= \frac{40\times 41\times 81}{6} - \frac{9\times 10\times 19}{6} = 21855$

$\therefore \left|\bar{x}+\bar{y}-\sigma^2\right| = |26+76-705| = 603$

Question ID : 366694355

24. Let $\{a_k\}$ and $\{b_k\}$, $k \in N$, be two G.P.s with common ratios r_1 and r_2 respectively such that $a_1 = b_1 = 4$ and $r_1 < r_2$. Let $c_k = a_k + b_k$, $k \in N$. If $c_2 = 5$ and $c_3 = \frac{13}{4}$ then $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$ is equal to ______ .

माना $\{a_k\}$ तथा $\{b_k\}$, $k \in N$, दो G.P.s है, जिनके सार्व अनुपात क्रमशः r_1 तथा r_2 है, और $a_1 = b_1 = 4$, $r_1 < r_2$ है। माना $c_k = a_k + b_k$, $k \in N$ है। यदि $c_2 = 5$ तथा $c_3 = \frac{13}{4}$ है, तो $\sum_{k=1}^{\infty} c_k - (12a_6 + 8b_4)$ बराबर है।

Ans. Official Answer NTA (9)

Sol. a = b = 4

$C_2 = a_2 + b_2 = 5$

$\Rightarrow ar_1 + br_2 = 5$

$\Rightarrow 4r_1 + 4r_2 = 5 \Rightarrow r_1 + r_2 = \frac{5}{4}$

$C_3 = a_3 + b_3 = \frac{13}{4} \Rightarrow ar_1^2 + br_2^2 = \frac{13}{4}$

$\Rightarrow 4r_1^2 + 4r_2^2 = \frac{13}{4}$

$\Rightarrow 4\left(\frac{5}{4} - r_2\right)^2 + 4r_2^2 = \frac{13}{4}$

$\Rightarrow 16\left(\frac{25}{16} - \frac{5}{2}r_2 + r_2^2\right) + 16r_2^2 = 13$

$\Rightarrow 25 - 40r_2 + 16r_2^2 + 16r_2^2 - 13 = 0$

$\Rightarrow 32r_2^2 - 40r_2 + 12 = 0$

$\Rightarrow 8r_2^2 - 10r_2 + 3 = 0$

$\Rightarrow (4r_2 - 3)(2r_2 - 1) = 0$

$\Rightarrow r_2 = \frac{3}{4} \text{ or } \frac{1}{2}$

$r_1 = \frac{1}{2} \text{ or } \frac{3}{4}$

$r_1 < r_2 \Rightarrow r_1 = \frac{1}{2}, r_2 = \frac{3}{4}$

$\sum_{k=1}^{\infty} C_k = \Sigma a_k + \Sigma b_k = \frac{a}{1-r_1} + \frac{b}{1-r_2} = \frac{4}{1-\frac{1}{2}} + \frac{4}{1-\frac{3}{4}}$

$= 8 + 16 = 24$

$\sum_{k=1}^{\infty} C_k - 12a_6 - 8b_4 = 24 - 12\left(4\left(\frac{1}{2}\right)^5\right) - 8.4.\left(\frac{3}{4}\right)^3$

$= 24 - \frac{3}{2} - \frac{27}{2} = 9$

Question ID : 366694352

25. Let $\alpha = 8 - 14i$, $A = \left\{z \in c : \frac{\alpha z - \bar{\alpha}\bar{z}}{z^2 - (\bar{z})^2 - 112i} = 1\right\}$ and $B = \{z \in c : |z + 3i| = 4\}$. Then $\sum_{z \in A \cap B} (\operatorname{Re} z - \operatorname{Im} z)$ is equal to ________ .

माना $\alpha = 8 - 14i$, $A = \left\{z \in c : \frac{\alpha z - \bar{\alpha}\bar{z}}{z^2 - (\bar{z})^2 - 112i} = 1\right\}$ तथा $B = \{z \in c : |z + 3i| = 4\}$ है। तो $\sum_{z \in A \cap B} (\operatorname{Re} z - \operatorname{Im} z)$

बराबर है।

Ans. Official Answer NTA (14)

Sol. $\alpha = 8 - 14i$

$z = x + iy$

$az = (8x + 14y) + i(-14x + 8y)$

$z + \bar{z} = 2x \qquad z - \bar{z} = 2iy$

Set A : $\dfrac{2i(-14x + 8y)}{i(4xy - 112)} = 1$

$(x - 4)(y + 7) = 0$

$x = 4$ or $y = -7$

Set B : $x^2 + (y + 3)^2 = 16$

when $x = 4$ $\qquad y = -3$

when $y = -7$ $\qquad x = 0$

$\therefore A \cap B = \{4 - 3i, 0 - 7i\}$

So, $\sum_{z \in A \cap B} (\text{Re}\, z - \text{Im}\, z) = 4 - (-3) + (0 - (-7)) = 14$

Question ID : 366694356

26. Let $a_1 = b_1$ and $a_n = a_{n-1} + (n - 1)$, $b_n = b_{n-1} + a_{n-1}$, $\forall n \geq 2$. If $S = \sum_{n=1}^{10} \frac{b_n}{2^n}$ and $T = \sum_{n=1}^{8} \frac{n}{2^{n-1}}$, then $2^7 (2S - T)$ is equal to ______ .

माना $a_1 = b_1$ तथा $a_n = a_{n-1} + (n - 1)$, $b_n = b_{n-1} + a_{n-1}$, $\forall n \geq 2$ है। यदि $S = \sum_{n=1}^{10} \frac{b_n}{2^n}$ तथा $T = \sum_{n=1}^{8} \frac{n}{2^{n-1}}$ हैं, तो $2^7 (2S - T)$ बराबर है।

Ans. Official Answer NTA (461)

Sol. $S = \dfrac{b_1}{2} + \dfrac{b_2}{2^2} + \dfrac{b_3}{2^3} + \ldots\ldots. + \dfrac{b_{10}}{2^{10}}$ (1)

$\dfrac{S}{2} = \dfrac{b_1}{2^2} + \dfrac{b_2}{2^3} + \dfrac{b_3}{2^4} + \ldots\ldots. + \dfrac{b_{10}}{2^{11}}$ (2)

$(1) - (2) \Rightarrow \dfrac{S}{2} = \dfrac{b_1}{2} + \dfrac{a_1}{2^2} + \dfrac{a_2}{2^3} + \ldots\ldots\ldots. + \dfrac{a_9}{2^{10}} - \dfrac{b_{10}}{2^{11}}$

$\Rightarrow S = b_1 + \dfrac{a_1}{2} + \dfrac{a_2}{2^2} + \ldots\ldots\ldots. + \dfrac{a_9}{2^9} - \dfrac{b_{10}}{2^{10}}$ (3)

$$\Rightarrow \frac{S}{2} = \frac{b_1}{2} + \frac{a_1}{2^2} + ... + \frac{a_9}{2^{10}} - \frac{b_{10}}{2^{11}} \qquad (4)$$

$$(2) - (4) \Rightarrow \frac{S}{2} = \frac{b_1}{2} + \frac{a_1}{2} + \frac{1}{2^2} + \frac{2}{2^3} .... + \frac{8}{2^9} - \frac{a_9}{2^{10}} - \frac{b_{10}}{2^{11}}$$

$$\Rightarrow \frac{S}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{2^2}\left(\frac{1}{1} + \frac{2}{2} + ... + \frac{8}{2^7}\right) - \frac{a_9}{2^{10}} - \frac{b_{10}}{2^{11}}$$

$$\Rightarrow \frac{S}{2} = 1 + \frac{1}{4}(T) - \frac{a_9}{2^{10}} - \frac{b_{10}}{2^{11}}$$

$$\Rightarrow 2S = 4 + T - \frac{a_9}{2^8} - \frac{b_{10}}{2^9}$$

$$\Rightarrow 2S - T = 4 - \frac{a_9}{2^8} - \frac{b_{10}}{2^9}$$

$$\Rightarrow 2^7(2S - T) = 2^9 - \frac{37}{2} - \frac{130}{2^2}$$

$= 461$

Question ID : 366694357

27. If the equation of the normal to the curve $y = \frac{x-a}{(x+b)(x-2)}$ at the point (1,–3) is $x - 4y = 13$, then the value of $a + b$ is equal to __________ .

यदि वक्र $y = \frac{x-a}{(x+b)(x-2)}$ के बिंदु (1,-3) पर अभिलंब का समीकरण $x - 4y = 13$ है, तो $x = \frac{5}{2}$ पर $a + b$ का मान बराबर है।

Ans. Official Answer NTA (4)

Sol. $y = \frac{x-a}{(x+b)(x-2)}$

At point (1, –3),

$$-3 = \frac{1-9}{(1+b)(1-2)}$$

$\Rightarrow 1 - a = 3(1 + b) \qquad ...(1)$

Now, $y = \frac{x-a}{(x+b)(x-2)}$

$\Rightarrow \frac{dy}{dx} = \frac{(x+b)(x-2)\times(1)-(x-a)(2x+b-2)}{(x+b)^2(x-2)^2}$

At $(1, -3)$ slope of normal is $\frac{1}{4}$ hence $\frac{dy}{dx} = -4$,

So, $-4 = \frac{(1+b)(-1)-(1-a)b}{(1+b)^2(-1)^2}$

Using equation (1)

$\Rightarrow -4 = \frac{(1+b)(-1)-3(b+1)b}{(1+b)^2}$

$\Rightarrow -4 = \left|\frac{(-1)-3b}{(1+b)}\right| (b \neq -1)$

$\Rightarrow b = -3$

So, $a = 7$

Hence, $a + b = 7 - 3 = 4$

Question ID : 366694353

28. Let A be the symmetric matrix such that $|A| = 2$ and $\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} A - \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$. If the sum of the diagonal elements of A is s, then $\frac{\beta s}{\alpha^2}$ is equal to ________ .

माना A एक सममित आव्यूह है जिसकी सारणिक 2 है तथा $\begin{bmatrix} 2 & 1 \\ 3 & \frac{3}{2} \end{bmatrix} A - \begin{bmatrix} 1 & 2 \\ \alpha & \beta \end{bmatrix}$ है। यदि A के विकर्ण के अवयवों का योग s है, तो $\frac{\beta s}{\alpha^2}$ बराबर है।

Ans. Official Answer NTA (5)

Sol. $A^T = A \Rightarrow$ So, $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

now $ad - b^2 = 2$ ...(1)

also $BA = \begin{bmatrix} 2 & 1 \\ 3 & 3/2 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix}$

$\Rightarrow 2a + b = 1$ ...(2)

$3a + \frac{3b}{2} = \alpha$ ...(3)

$3b + \frac{3d}{2} = \beta$ ...(4)

$2b + d = 2$ ...(5)

Solving equations

$a = \frac{3}{4},\ \alpha = \frac{3}{2},\ b = -\frac{1}{2}$

$d = 3,\ \beta = 3$

So $\frac{\beta \times s}{\alpha} = 5$

Question ID : 366694358

29. A circle with centre (2,3) and radius 4 intersects the line $x+y=3$ at the points P and Q. If the tangents at P and Q intersect at the point $S(\alpha,\beta)$, then $4\alpha - 7\beta$ is equal to ________.

एक वृत्त, जिसका केन्द्र (2,3) है तथा त्रिज्या 4 है, रेखा $x+y=3$ को बिंदुओं P तथा Q पर काटता है। यदि P तथा Q पर स्पर्श रेखाएँ बिंदु $S(\alpha,\beta)$ पर मिलती है, तो $4\alpha - 7\beta$ बराबर है।

Ans. Official Answer NTA (11)

Sol. The given line is polar or P(2, β) w.r.t. given circle

$x^2 + y^2 - 4x - 6y - 3 = 0$

Chord or contact

$\alpha x + \beta y - 2(x + \alpha) - 3(y + \beta) - 3 = 0$

$\Rightarrow (\alpha - 2)x + (\beta - 3)y - (2\alpha + 3\beta + 3) = 0$ ...(i)

$\because$ But the equation of chord of contact is given

as : $x + y - 3 = 0$ ...(ii)

comparing the coefficients

$\frac{\alpha - 2}{1} = \frac{\beta - 3}{1} = -\left(\frac{2\alpha + 3\beta + 3}{-3}\right)$

On solving $\alpha = -6$

$\beta = -5$

Now $4\alpha - 7\beta = 11$

Question ID : 366694354

30. The total number of 4-digit numbers whose greatest common divisor with 54 is 2, is ______ .

4-अंकों की संख्याओं, जिनका 54 के साथ महत्तम उभयनिष्ठ भाजक 2 है, की कुल संख्या है।

Ans. Official Answer NTA (3000)

Sol. Four digit numbers which are divisible by 2 $\Rightarrow$ 1000 to 9998 $\Rightarrow$ 4500

Four digit numbers divisible by 6

$\Rightarrow$ 1002 to 9996 $\Rightarrow$ 1500

Four digit integer divisible by 2 but not divisible by 3 $\Rightarrow 4500 - 1500 = 3000$