JEE Main July 2021

Question Paper With Text Solution

27 July. | Shift-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JULY 2021 | 27TH JULY SHIFT-1

SECTION – A

1. The probability that a randomly selected 2-digit number belongs to the set $\{n \in N : (2^n - 2)$ is a multiple of $3\}$ is equal to :

(1) 2/3 (2) 1/2 (3) 1/3 (4) 1/6

Ans. Official Answer NTA (2)

Sol. Total two digit numbers = 90

$2^n - 2 = (3 - 1)^n - 2$

$= {}^nC_0\, 3^n - {}^nC_1\, 3^{n-1} + {}^nC_2\, 3^{n-2} + \ldots\ldots + {}^nC_n\, (-1)^n - 2$

$= 3\,({}^nC_0\, 3^{n-1} - {}^nC_1\, 3^{n-2} \ldots\ldots\ldots) + (-1)^n - 2$

2n –2 is a multiple of 3 of n is odd.

favourable = 45

$P = \frac{45}{90} = \frac{1}{2}$

2. Let

$A = \{(x, y) \in R \times R \mid 2x^2 + 2y^2 - 2x - 2y = 1\}$,

$B = \{(x, y) \in R \times R \mid 4x^2 + 4y^2 - 16y + 7 = 0\}$ and

$C = \{(x, y) \in R \times R \mid x^2 + y^2 - 4x - 2y + 5 \le r^2\}$.

Then the minimum value of $|r|$ such that $A \cup B \subseteq C$ is equal to :

(1) $\frac{3 + 2\sqrt{5}}{2}$ (2) $\frac{2 + \sqrt{10}}{2}$ (3) $1 + \sqrt{5}$ (4) $\frac{3 + \sqrt{10}}{2}$

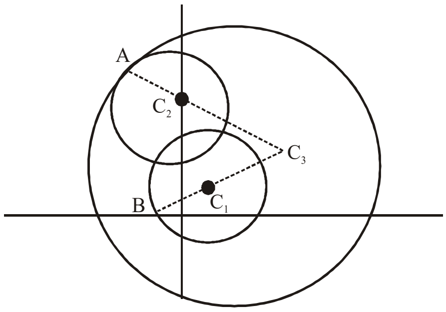
Ans. Official Answer NTA (1)

Sol. $A : x^2 + y^2 - x - y - \frac{1}{2} = 0;\ C_1 = \left(\frac{1}{2}, \frac{1}{2}\right),\ r_1 = 1$

$B : x^2 + y^2 - 4y + \frac{7}{4} = 0;\ C_2 = (0, 2),\ r_2 = \frac{3}{2}$

$C : x^2 + y^2 - 4x - 2y + 5 - r^2 \le 0;\ C_3 = (2, 1),\ r_3 = |r|$

If $A \cup B \subseteq C$



$r_3 \geq C_2 C_3 + r_2$ $\quad\quad$ $(C_2C_3 > C_1C_3)$

$|r| \geq \sqrt{5} + \frac{3}{2}$

$|r| \geq \frac{3+2\sqrt{5}}{2}$

3. The compound statement $(P \vee Q) \wedge (\sim P) \Rightarrow Q$ is equivalent to :

(1) $\sim (P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$ $\quad$ (2) $P \wedge \sim Q$

(3) $\sim (P \Rightarrow Q)$ $\quad$ (4) $P \vee Q$

Ans. Official Answer NTA (1)

Sol. $(P \vee Q) \wedge (\sim P) \Rightarrow Q$

$= (P \wedge \sim P) \vee (Q \wedge \sim P) \Rightarrow Q$ $\quad$ $\{(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)\}$

$= c \vee (Q \wedge \sim P) \Rightarrow Q$

$= (Q \wedge \sim P) \Rightarrow Q$

$= \sim (Q \wedge \sim P) \vee Q$ $\quad$ $\{ p \rightarrow q = \sim p \vee q \}$

$= (\sim Q \vee P) \vee Q$ $\quad$ $\{\sim (p \wedge q) = \sim p \vee \sim q\}$

$= (\sim Q \vee Q) \vee P$

$= t \vee P$

$= t$

Option 1 :

$= \sim (P \Rightarrow Q) \Leftrightarrow P \wedge \sim Q$

$= \sim (\sim P \vee Q) \Leftrightarrow P \wedge \sim Q$

$= P \wedge \sim Q \Leftrightarrow P \wedge \sim Q$

$= t$

4. Let the plane passing through the point (–1, 0, – 2) and perpendicular to each of the planes $2x + y - z = 2$ and $x - y - z = 3$ be $ax + by + cz + 8 = 0$. Then the value of $a + b + c$ is equal to :

(1) 5 (2) 8 (3) 3 (4) 4

Ans. Official Answer NTA (4)

Sol. A (–1, 0, – 2)

P_1; $2x + y - z = 2$; $\vec{n}_1 = 2i + j - k$

P_2; $x - y - z = 3$; $\vec{n}_2 = i - j - k$

$\vec{n} = \vec{n}_1 \times \vec{n}_2$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = -2i + j - 3k$$

DR's of normal = 2, –1, 3

Equation of plane $\Rightarrow 2(x + 1) -1 (y - 0) + 3(z + 2) = 0$

$\Rightarrow 2x - y + 3z + 8 = 0$

$a + b + c = 2 - 1 + 3 = 4$

5. Two tangents are drawn from the point P (–1, 1) to the circle $x^2 + y^2 - 2x - 6y + 6 = 0$. If these tangents touch the circle at points A and B, and if D is a point on the circle such that length of the segments AB and AD are equal, then the area of the triangle ABD is equal to :

(1) $3(\sqrt{2}-1)$ (2) 2 (3) $(3\sqrt{2}+2)$ (4) 4

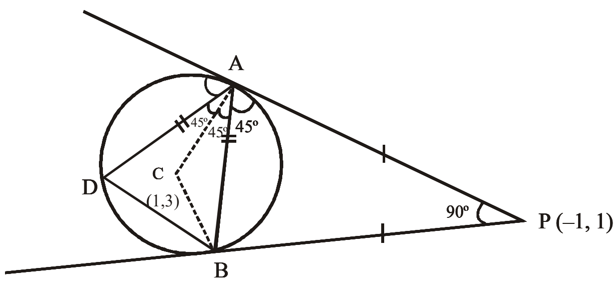
Ans. Official Answer NTA (4)

Sol. P (–1, 1)

C : $x^2 + y^2 - 2x - 6y + 6 = 0$

C: $(x - 1)^2 + (y - 3)^2 = 2^2$

$PA = \sqrt{S_1} = 2$



$PC = 2\sqrt{2} \Rightarrow$ P lies on the director circle $(\angle APB = 90^\circ)$

$AB = 2\sqrt{2}$

Area of $\Delta ADB = \frac{1}{2} \times AD \times AB$

$= \frac{1}{2} \times 2\sqrt{2} \times 2\sqrt{2}$

$= 4$

6. If $\sin\theta + \cos\theta = \frac{1}{2}$, then $16\left(\sin(2\theta)+\cos(4\theta)+\sin(6\theta)\right)$ is equal to :

(1) 27 (2) – 27 (3) 23 (4) – 23

Ans. Official Answer NTA (4)

Sol. $\sin\theta + \cos\theta = \frac{1}{2} \Rightarrow \sin2\theta = \frac{-3}{4}$

$16(\sin 2\theta + \cos4\theta + \sin 6\theta)$

$= 16(\sin 2\theta + (1 - 2\sin^2 2\theta) + (3\sin2\theta - 4\sin^3 2\theta))$

$= 16(1 + 4\sin2\theta - 2\sin^2 2\theta - 4\sin^3 2\theta)$

$= -23$

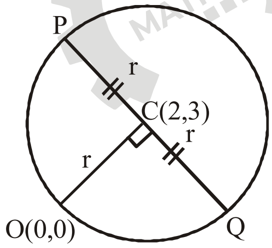
7. Let P and Q be two distinct points on a circle which has center at C(2,3) and which passes through origin O. If OC is perpendicular to both the line segments CP and CQ, then the set {P, Q} is equal to.

(1) $\left\{\left(2+2\sqrt{2},3+\sqrt{5}\right),\left(2-2\sqrt{2},3-\sqrt{5}\right)\right\}$ (2) $\left\{\left(2+2\sqrt{2},3-\sqrt{5}\right),\left(2-2\sqrt{2},3+\sqrt{5}\right)\right\}$

(3) {(4,0), (0, 6)} (4) {(–1, 5), (5,1)}

Ans. Official Answer NTA (4)

Sol.



$m_{OC} = \frac{3}{2}$

$m_{PQ} = \frac{-2}{3}$

$\sin\theta = \frac{2}{\sqrt{13}}, \cos\theta = \frac{-3}{\sqrt{13}}$

Equation of PQ $\frac{x-2}{\cos\theta} = \frac{y-3}{\sin\theta} = r$

Coordinates of point P & Q

$$= \frac{\frac{x-2}{\frac{-3}{\sqrt{13}}}}{} = \frac{y-3}{\frac{2}{\sqrt{13}}} = \pm\sqrt{13}$$

$= (2 \mp 3, 3 \pm 2)$

$= (-1, 5)$ and $(5, 1)$

8. Let α, β be two roots of the equation $x^2 + (20)^{1/4} x + (5)^{1/2} = 0$. Then $\alpha^8 + \beta^8$ is equal to -

(1) 100 (2) 10 (3) 160 (4) 50

Ans. Official Answer NTA (4)

Sol. $x^2 + (20)^{1/4} + (5)^{1/2} = 0$

$\alpha + \beta = -(20)^{1/4}$

$\alpha . \beta = 5^{1/2}$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^2 + \beta^2 = 20^{1/2} - 2.5^{1/2} = 0$

$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = -10$

$\alpha^8 + \beta^8 = (\alpha^4 + \beta^4)^2 - 2\alpha^4\beta^4 = 50$

9. If the mean and variance of the following data:

6, 10, 7, 13, a, 12, b, 12

are 9 and 37/4 respectively, then $(a - b)^2$ is equal to :

(1) 12 (2) 24 (3) 16 (4) 32

Ans. Official Answer NTA (3)

Sol. $\bar{x} = \frac{6+10+7+13+a+12+b+12}{8} = 9$

$a + b = 12$

$\sigma^2 = \frac{\sum xi^2}{8} - (\bar{x})^2 = \frac{37}{4}$

$\Rightarrow \frac{6^2+10^2+7^2+13^2+a^2+12^2+b^2+12^2}{8} - 81 = \frac{37}{4}$

$\Rightarrow a^2 + b^2 = 80$

$(a - b)^2 + (a + b)^2 = 2(a^2 + b^2)$

$(a - b)^2 = 160 - 144 = 16$

10. If the coefficients of x^7 in $\left(x^2+\frac{1}{bx}\right)^{11}$ and x^{-7} in $\left(x-\frac{1}{bx^2}\right)^{11}$, $b \neq 0$, are equal, then the value of b is equal to :

(1) -2 (2) -1 (3) 2 (4) 1

Ans. Official Answer NTA (4)

Sol. $\left(x^2+\frac{1}{bx}\right)^{11} \Rightarrow T_{r+1} = {}^{11}C_r (x^2)^{11-r} \left(\frac{1}{bx}\right)^r$

$$T_{r+1} = {}^{11}C_r \frac{1}{b^r} x^{22-3r}$$

$$\text{Coefficient of } x^7 = {}^{11}C_5 \cdot \frac{1}{b^5}$$

$\left(x-\frac{1}{bx^2}\right)^{11} \Rightarrow T_{r+1} = {}^{11}C_r x^{11-r} \left(\frac{-1}{bx^2}\right)^r$

$$T_{r+1} = {}^{11}C_r \left(\frac{-1}{b}\right)^r x^{11-3r}$$

Coefficient of $x^{-7} = {}^{11}C_6 \cdot \frac{1}{b^6}$

${}^{11}C_5 \frac{1}{b^5} = {}^{11}C_6 \frac{1}{b^6} \Rightarrow b = 1$

11. The value of is equal to $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{(2j-1)+8n}{(2j-1)+4n}$ is equal to :

(1) $5+\log_e\left(\frac{3}{2}\right)$ (2) $1+2\log_e\left(\frac{3}{2}\right)$ (3) $3+2\log_e\left(\frac{2}{3}\right)$ (4) $2-\log_e\left(\frac{2}{3}\right)$

Ans. Official Answer NTA (2)

Sol. $\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{2\frac{j}{n}-\frac{1}{n}+8}{2\frac{j}{n}-\frac{1}{n}+4} = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{2\frac{j}{n}+8}{2\frac{j}{n}+4}$ $\left(\lim_{n\to\infty}\frac{1}{n}=0\right)$

$$= \int_0^1 \frac{2x+8}{2x+4}dx = \int_0^1\left(1+\frac{4}{2x+4}\right)dx = x+\ell n(x+2)\Big|_0^1$$

$$= 1 + 2\, ln\left(\frac{3}{2}\right)$$

12. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. If $A^{-1} = \alpha I + \beta A$, $\alpha, \beta \in R$, I is a 2×2 idently matrix, then $4(\alpha - \beta)$ is equal to.

(1) 2 (2) 4 (3) 8/3 (4) 5

Ans. Official Answer NTA (2)

Sol. $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

tr (A) = 5

$|A| = 6$

$A^2 - 5A + 6I = 0$ (Characteristic equation)

$A^{-1} = \alpha I + \beta A$

$AA^{-1} = \alpha AI + \beta AA$

$\beta A^2 + \alpha A - I = 0$

$\frac{1}{\beta} = \frac{-5}{\alpha} = \frac{6}{-1} \Rightarrow \alpha = \frac{5}{6},\ \beta = \frac{-1}{6} \quad \Rightarrow \quad 4(\alpha - \beta) = 4$

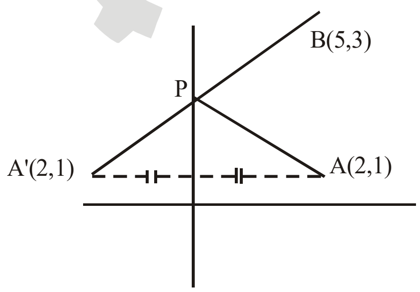
13. A ray of light through (2, 1) is reflected at a point P on the y-axis and then passes through the point (5,3). If this reflected ray is the directrix of an ellipse with eccentricity 1/3 and the distance of the nearer focus from this directrix is $\frac{8}{\sqrt{53}}$, then the equation of the other directrix can be :

(1) $11x + 7y + 8 = 0$ or $11x + 7y - 15 = 0$ (2) $11x - 7y - 8 = 0$ or $11x + 7y + 15 = 0$

(3) $2x - 7y - 39 = 0$ or $2x - 7y - 7 = 0$ (4) $2x - 7y - 29 = 0$ or $2x - 7y - 7 = 0$

Ans. Official Answer NTA (4)

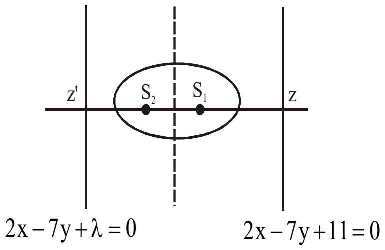
Sol.



$m_{A'B} = \frac{3-1}{5+2} = \frac{2}{7}$

Equation of A'B $\Rightarrow (y - 1) = \frac{2}{7}(x + 2)$

$\Rightarrow 2x - 7y + 11 = 0$



$e = \frac{1}{3}$

$S_1Z = \frac{8}{\sqrt{53}} = \frac{a}{e} - ae = 3a - \frac{a}{3} = \frac{8a}{3}$

$a = \frac{8}{\sqrt{53}}$

$ZZ' = 2\frac{a}{e} = \frac{18}{\sqrt{53}}$

$\Rightarrow \left|\frac{\lambda - 11}{\sqrt{53}}\right| = \frac{18}{\sqrt{53}}$

$\Rightarrow \lambda - 11 = \pm 18$

$\Rightarrow \lambda = 29$ or -7

Equation of other directrix

$2x - 7y + 29 = 0$

or $2x - 7y - 7 = 0$

14. Let $y = y(x)$ be solution of the differentiable equation $\log_e\left(\frac{dy}{dx}\right) = 3x + 4y$, with $y(0) = 0$.

If $y\left(-\frac{2}{3}\log_e 2\right) = \alpha \log_e 2$, then the value of α is equal to -

(1) 2 (2) $-\frac{1}{4}$ (3) $\frac{1}{4}$ (4) $-\frac{1}{2}$

Ans. Official Answer NTA (2)

Sol. $\frac{dy}{dx} = e^{3x+4y}$

$\Rightarrow \int e^{-4y} dy = \int e^{3x}\, dx$

$\Rightarrow \frac{-e^{-4y}}{4} = \frac{e^{3x}}{3} + C$ (y (0) = 0)

$\frac{-1}{4} = \frac{1}{3} + C \Rightarrow C = \frac{-7}{12}$

$\Rightarrow \frac{-e^{4y}}{4} = \frac{e^{3x}}{3} - \frac{7}{12}$

$\Rightarrow -3e^{-4y} = 4e^{3x} - 7$

$\Rightarrow e^{-4y} = \frac{7-4e^{3x}}{3}$

$y = \frac{-1}{4}\, \ln\left(\frac{7-4e^{3x}}{3}\right)$

$y\left(\frac{-2}{3}\log_e 2\right) = \frac{1}{4}\, \ln\left(\frac{7-4e^{3\left(\frac{-2}{3}\log_e 2\right)}}{3}\right)$

$= \frac{-1}{4}\, \ln 2$

15. The value of definite integral $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{x\cos x}\right)\left(\sin^4 x + \cos^4 x\right)}$ is equal to :

(1) $-\frac{\pi}{4}$ (2) $\frac{\pi}{\sqrt{2}}$ (3) $-\frac{\pi}{2}$ (4) $\frac{\pi}{2\sqrt{2}}$

Ans. Official Answer NTA (4)

Sol. $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{x\cos x}\right)\left(\sin^4 x + \cos^4 x\right)}$ ....(1)

Using $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{x\cos x}\right)\left(\sin^4 x + \cos^4 x\right)}$ ....(2)

$2I = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x} = 2\int_0^{\frac{\pi}{4}} \frac{dx}{\sin^4 x + \cos^4 x}$

$I = \int_0^{\frac{\pi}{4}} \frac{\left(1+\tan^2 x\right)\sec^2 x}{\left(1+\tan^4 x\right)} dx$

tan x = t

$\sec^2 x\, dx = dt$

$$I = \int_0^1 \frac{1+t^2}{1+t^4}dt = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right)}{t^2+\frac{1}{t^2}}dt$$

$$I = \int_0^1 \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+2}$$

$$I = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)\Bigg|_0^1 = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)\Bigg|_0^1 = \frac{\pi}{2\sqrt{2}}$$

16. Let C be the set of all complex numbers. Let

$S_1 = \{z \in C \mid |z-3-2i|^2 = 8\}$,

$S_2 = \{z \in C \mid Re(z) \geq 5\}$ and

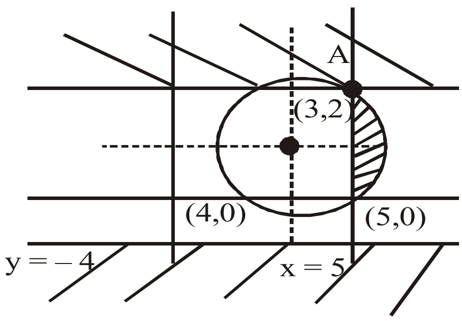
$S_3 = \{z \in C \mid |z-\bar{z}| \geq 8\}$.

Then the number of elements is $S_1 \cap S_2 \cap S_3$ is equal to :

(1) 1 (2) 0 (3) 2 (4) Infinite

Ans. Official Answer NTA (1)

Sol. $S_1 : |z-(3+2i)| = 2\sqrt{2}$



$S_2 : x \geq 5$

$S_3 : |y| \geq 4$

$S_1 \cap S_2 \cap S_3 = A$

17. Let $f:\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\to R$ be defined as

$$f(x)=\begin{cases}(1+|\sin x|)^{\frac{3a}{|\sin x|}} & , -\frac{\pi}{4}<x<0\\ b & , x=0\\ e^{\cot 4x/\cot 2x} & , 0<x<\frac{\pi}{4}\end{cases}.$$

If f is continuous at x = 0, then the value of $6a + b^2$ is equal to :

(1) e – 1 (2) 1 – e (3) e (4) 1 + e

Ans. Official Answer NTA (4)

Sol. $f(0) = f(0^+) = f(0^-)$

$$b=\lim_{x\to 0^+}(1+|\sin x|)^{\frac{3a}{|\sin x|}}=\lim_{x\to 0^-}e^{\frac{\cot 4x}{\cot 2x}}$$

$$b=e^{\lim_{x\to 0^+}\frac{3a}{|\sin x|}(1+|\sin x|-1)}=\lim_{x\to 0^-}e^{\frac{\tan 2x}{\tan 4x}}$$

$$b=e^{3a}=e^{1/2}\Rightarrow a=\frac{1}{6},\ b=\sqrt{e}$$

$6a + b^2 = 1 + e$

18. If the area of the bounded region

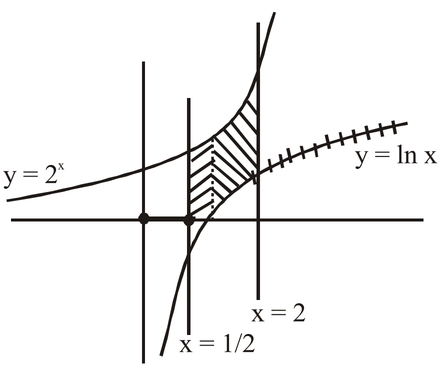
$$R=\left\{(x,y):\max\{0,\log_e x\}\le y\,2^x,\frac{1}{2}\le x\le 2\right\}$$

is $\alpha(\log_e 2)^{-1}+\beta(\log_e 2)+\gamma$, then value of $(\alpha+\beta-2\gamma)^2$ is equal to :

(1) 4 (2) 8 (3) 1 (4) 2

Ans. Official Answer NTA (4)

Sol.



$$A = \int_{\frac{1}{2}}^{1} 2^x dx + \int_{1}^{2} \left(2^x - \ln x\right) dx$$

$$A = \left.\frac{2^x}{\ln 2}\right|_{\frac{1}{2}}^{2} + \left.\frac{2^x}{\ln 2} - (x\ln x - x)\right|_{1}^{2}$$

$$A = \left(4-\sqrt{2}\right)\log_2 e - 2\log_2 e + 1$$

$$\alpha = 4-\sqrt{2}, \beta = -2, \gamma = 1$$

$(\alpha + \beta - 2\gamma)^2 = 2$

19. Let $f: R \to R$ be a function such that $f(2) = 4$ and $f'(2) = 1$. Then, the value of

$\lim\limits_{x\to 2}\dfrac{x^2 f(2) - 4f(x)}{x-2}$ is equal to

(1) 16 (2) 8 (3) 12 (4) 4

Ans. Official Answer NTA (3)

Sol. $\lim\limits_{x\to 2}\dfrac{x^2 f(2) - 4f(x)}{x-2}$ $\left(\dfrac{0}{0}\right)$

Use L'Hospital Rule

$$\lim_{x\to 2}\frac{2x\ f(2) - 4f'(x)}{1}$$

$\Rightarrow 4f(2) - 4f'(2) = 12$

20. Let $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + 3\hat{k}$. Then the vector product $\left(\vec{a}+\vec{b}\right) \times \left(\left(a \times \left(\left(\vec{a}-\vec{b}\right)\times\vec{b}\right)\right)\times\vec{b}\right)$ is equal to.

(1) $7\left(30\hat{i} - 5\hat{j} + 7\hat{k}\right)$ (2) $7\left(34\hat{i} - 5\hat{j} + 3\hat{k}\right)$ (3) $5\left(30\hat{i} - 5\hat{j} + 7\hat{k}\right)$ (4) $5\left(34\hat{i} - 5\hat{j} + 3\hat{k}\right)$

Ans. Official Answer NTA (2)

Sol. $\left(\vec{a}+\vec{b}\right) \times \left(\left(\vec{a} \times \left(\left(\vec{a}-\vec{b}\right)\times\vec{b}\right)\right)\times\vec{b}\right)$

$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(\left(\vec{a}\times\left(\vec{a}\times b\right)\right)\times\vec{b}\right)$

$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(\left(\left(\vec{a}.\vec{b}\right)\vec{a} - \left(\vec{a}.\vec{a}\right)\vec{b}\right)\times\vec{b}\right)$

$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(\left(7\vec{a} - 6\vec{b}\right)\times\vec{b}\right)$

$\Rightarrow \left(\vec{a}+\vec{b}\right)\times\left(7\left(\vec{a}\times\vec{b}\right)\right)$

$\Rightarrow 7\left\{\vec{a}\times\left(\vec{a}\times\vec{b}\right)+\vec{b}\times\left(\vec{a}\times\vec{b}\right)\right\}$

$\Rightarrow 7\left\{\left(\vec{a}.\vec{b}\right)\vec{a}-\left(\vec{a}.\vec{a}\right)\vec{b}+\left(\vec{b}.\vec{b}\right)\vec{a}-\left(\vec{b}.\vec{a}\right)\vec{b}\right\}$

$\Rightarrow 7\left\{7\vec{a}-6\vec{b}+14\vec{a}-7\vec{b}\right\}$

$\Rightarrow 7\left\{21\vec{a}-13\vec{b}\right\}$

$\Rightarrow 7\left\{21\left(i+j+2k\right)-13(-i+2j+3k)\right\}$

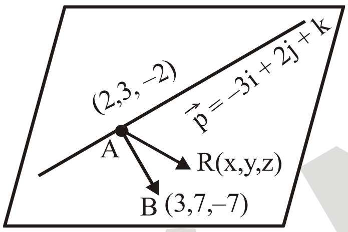
$\Rightarrow 7\ (34i - 5j + 3k)$

Section B

1. Let a plane P pass through the point (3, 7, –7) and contain the line $\frac{x-2}{-3}=\frac{y-3}{2}=\frac{z+2}{1}$. If distance of the plan P from the originis d, then d^2 is equal to.

Ans. Official Answer NTA (3)

Sol.



$\overline{AR}, \overline{AB}, \vec{P}$ are collinear vectors

$$\begin{vmatrix} x-2 & y-3 & z+2 \\ 1 & 4 & -5 \\ -3 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x-2)(14) - (y-3)(-14) + (z+2)(14) = 0$

$\Rightarrow x + y + 7 - 3 = 0$

$d = \left|\frac{0+0+0-3}{\sqrt{3}}\right| = \sqrt{3}$

$d^2 = 3$

2. Let $f(x) = \begin{vmatrix} \sin^2 x & -2+\cos^2 x & \cos 2x \\ 2+\sin^2 x & \cos^2 x & \cos 2x \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}$, $x \in [0, \pi]$.

Then the maximum value of f(x) is equal to________.

Ans. Official Answer NTA (6)

Sol. $f(x) = \begin{vmatrix} \sin^2 x & -2+\cos^2 x & \cos^2 x \\ 2+\sin^2 x & \cos^2 x & \cos^2 x \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}$

$R_1 \to R_1 - R_2,\ R_2 \to R_2 - R_3$

$f(x) = \begin{vmatrix} -2 & -2 & 0 \\ 2 & 0 & -1 \\ \sin^2 x & \cos^2 x & 1+\cos 2x \end{vmatrix}$

$f(x) = 2\sin^2 x - 2\cos^2 x + (1 + \cos 2x).4$

$f(x) = 2\cos 2x + 4$

$f(x)_{max} = 2 + 4 = 6$

3. For real numbers α and β consider the following system of linear equations :

$x + y - z = 2,\ x + 2y + \alpha z = 1,\ 2x - y + z = \beta$.

If the system has infinite solutions, then $\alpha + \beta$ is equal to.

Ans. Official Answer NTA (5)

Sol. $x + y - z = 2$ ....(i)

$x + 2y + \alpha z = 1$ ....(ii)

$2x - y + z = \beta$ ....(iii)

(ii) – (i) $\Rightarrow$ $y + (\alpha + 1) z = -1$

(iii) – 2 (ii) $\Rightarrow$ $-5y + (1 - 2\alpha)z = \beta - 2$

For infinite solution

$\frac{1}{-5} = \frac{\alpha+1}{1-2\alpha} = \frac{-1}{\beta-2}$

$\alpha = -2$

$\beta = 7$

$\alpha + \beta = 5$

4. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b}$ and $\vec{c} = \hat{j} - \hat{k}$ be three vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a} \cdot \vec{b} = 1$. If the length of projection vector of the vector $\vec{b}$ on the vector $\vec{a} \times \vec{c}$ is l, then the value of $3l^2$ is equal to ________.

Ans. Official Answer NTA (2)

Sol. $l = \left|\dfrac{\vec{b}.(\vec{a} \times \vec{c})}{|\vec{a} \times \vec{c}|}\right| = \left|\dfrac{\left[\vec{b}\,\vec{a}\,\vec{c}\right]}{|\vec{a} \times \vec{c}|}\right| = \left|\dfrac{\left[\vec{a}\ \vec{b}\ \vec{c}\right]}{|\vec{a} \times \vec{c}|}\right|$

$$\vec{a} \times \vec{c} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -2i + j + k$$

$\vec{a} \times \vec{b} = \vec{c}$

$\left(\vec{a} \times \vec{b}\right).\vec{c} = \vec{c}.\vec{c} \Rightarrow \left[\vec{a}\ \vec{b}\ \vec{c}\right] = 2$

$l = \dfrac{2}{\sqrt{6}} = \sqrt{\dfrac{2}{3}}$

$3l^2 = 2$

5. Let S = {1, 2, 3,4, 5, 6, 7}. Then number of possible functions $f : S \to S$ such that $f(m \cdot n) = f(m) \cdot f(n)$ for every m, n $\in$ S and m·n $\in$ S is equal to_______.

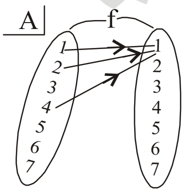
Ans. Official Answer NTA (490)

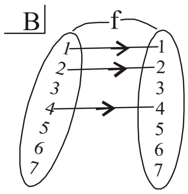
Sol. $f(m.n) = f(m).\ f(n)$ for $\forall$ m.n $\in$ S

$m = 1$: $f(x) = f(1).\ f(x) \Rightarrow f(x) = 1$

$m = n = 2$: $f(4) = f^2(2) \Rightarrow$ $f(2) = 1$ & $f(4) = 1$

$f(2) = 2$ & $f(4) = 4$





$m = 2, n = 3$: $f(6) = f(2).\ f(3)$

for A | $f(2) = 1 \Rightarrow f(6) = f(3) \in \{1,2,3,4,5,6,7\}$

$f(5) = f(7) \in \{1,2,3,4,5,6,7\}$ (can take any value)

Total = 7 × 7 × 7 = 343

for B | $f(2) = 2 \Rightarrow f(6) = 2\ f(3)$

f(3)	f(6)
1	2
2	4
3	6

Total = 3 × 7 × 7 = 147

Total functions = 343 + 147 = 490

6. If y = y(x), $y \in \left[0, \frac{\pi}{2}\right)$ is the solution of the differential equation

$\sec y \frac{dy}{dx} - \sin(x + y) - \sin(x - y) = 0$, with y(0) = 0, then $5y'\left(\frac{\pi}{2}\right)$ is equal to_______.

Ans. Official Answer NTA (2)

Sol. $\sec y \frac{dy}{dx} = 2\sin x \cos y$

$\Rightarrow \int \sec^2 y\, dy = \int 2\sin x\, dx$

$\Rightarrow \tan y = -2\cos x + C$

$y(0) = 0 \Rightarrow C = 2$

$\tan y = 2 - 2\cos x$

at $x = \frac{\pi}{2}$ $\tan y = 2$

$\sec^2 y \frac{dy}{dx} = 2\sin x \Rightarrow \frac{dy}{dx} = \frac{2\sin x}{1 + \tan^2 y}$

$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{2}} = \frac{2}{1+4} = \frac{2}{5}$

$5y'\left(\frac{\pi}{2}\right) = 2$

7. If $\log_3 2$, $\log_3(2^x - 5)$, $\log_3\left(2^x - \frac{7}{2}\right)$ are in an arithmetic progression, then the value of x is equal to.

Ans. Official Answer NTA (3)

Sol. $2\log_3(2^x - 5)$ $\log_3 2 + \log_3\left(2x - \frac{7}{2}\right)$

$(2^x - 5)^2 = 2\left(2^x - \frac{7}{2}\right)$

$(t - 5)^2 = 2t - 7$

$\Rightarrow t^2 - 12t + 32 = 0$

$2^x = 4, 8$

x = 2, 3 (for x = 2, $2^x - 5 < 0$)

So x = 3

8. Let the domain of the function

$f(x) = \log_4\left(\log_5\left(\log_3\left(18x - x^2 - 77\right)\right)\right)$ be (a, b).

Then the value of the integral

$\int_a^b \frac{\sin^3 x}{\left(\sin^3 x + \sin^3(a+b-x)\right)} dx$ is equal to.

Ans. Official Answer NTA (1)

Sol. $\log_5 (\log_3 (18x - x^2 - 77) > 0$

$\Rightarrow \log_3 (18x - x^2 - 77) > 1$

$\Rightarrow 18x - x^2 - 77 > 3$

$\Rightarrow x^2 - 18x + 80 < 0$

$x \in (8, 10)$

$I = \int_8^{10} \frac{\sin^3 x}{\sin^3 x + \sin^3(18-x)} dx$ .... (i)

Use $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$I = \int_8^{10} \frac{\sin^3 x(18-x)}{\sin^3(18-x) + \sin^3 x} dx$ .... (ii)

(i) + (ii)

$2I = \int_8^{10} dx = 2$

$I = 1$

9. Let f: $[0,3] \to R$ be defined by

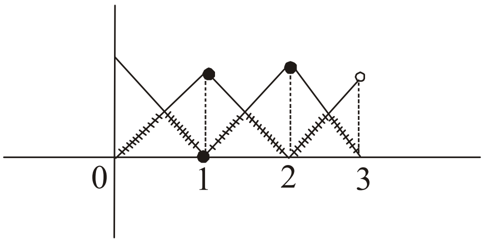
$f(x) = \min\{x - [x], 1 + [x] - x\}$

where [x] is the greatest integer less than or equal to x.

Let P denote the set containing all $x \in [0,3]$ where *f* is discontinuous, and Q denote the set containing all $x \in (0,3)$ where *f* is is not differentiable. Then the sum of number of elements in P and Q is equal to________.

Ans. Official Answer NTA (5)

Sol. $f(x) = \min\left(\{x\}, 1 - \{x\}\right)$



n(P) = 0
n(Q) = 5
n (P) + n (Q) = 5

10. Let $F : [3, 5] \to R$ be a twice differentiable function on (3,5) such that $F(x) = e^{-x}\int_3^x \left(3t^2 + 2t + 4F'(t)\right)dt$.

If $F'(4) = \dfrac{\alpha e^{\beta} - 224}{\left(e^{\beta} - 4\right)^2}$, then $\alpha + \beta$ is equal to________.

Ans. Official Answer NTA (16)

Sol. $e^x F(x) = \int_3^x \left(3t^2 + 2t + 4F'(t)\right)dt$ ....(1)

$e^x F'(x) + e^x F(x) = 3x^2 + 2x + 4F'(x)$

Put x = 4

$e^4 F'(4) + e^4 F(4) = 48 + 8 + 4F'(4)$

$F'(4) = \dfrac{56 - e^4 F(4)}{e^4 - 4}$

Now put x = 4 in equation(1)

$e^4 F(4) = \int_3^4 \left(3t^2 + 2t + 4F'(t)\right)dt$

$e^4 F(4) = t^3 + t^2 + 4F(t)\Big|_3^4$

$e^4 F(4) = (4^3 - 3^3) + (4^2 - 3^2) + 4[F(4) - F(3)]$ (F(3) = 0)

$(e^4 - 4) F(4) = 44$

$F'(4) = \dfrac{56 - e^4\left(\dfrac{44}{e^4 - 4}\right)}{\left(e^4 - 4\right)} = \dfrac{12e^4 - 224}{\left(e^4 - 4\right)^2}$

$\alpha = 12, \beta = 4 \Rightarrow \alpha + \beta = 16$