JEE Main July 2021

Question Paper With Text Solution

27 July. | Shift-2

MATHEMETICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JULY 2021 | 27TH JULY SHIFT-2

SECTION – A

1. Let $\alpha = \max\limits_{x\in R}\left\{8^{2\sin 3x}.4^{4\cos 3x}\right\}$ and $\beta = \min\limits_{x\in R}\left\{8^{2\sin 3x}\cdot 4^{4\cos 3x}\right\}$.

If $8x^2 + bx + c = 0$ is a quadratic equation whose roots are $\alpha^{1/5}$ and $\beta^{1/5}$, then the value of c – b is euqal to :

(1) 47 (2) 50 (3) 43 (4) 42

Ans. Official Answer NTA (4)

Sol. $\alpha = \max\limits_{x\in R}\{2^{6\sin 3x + 8\cos 3x}\}$

$= \max\limits_{x\in R}\{2^{[-10,10]}\}$

$\alpha = 2^{10}$

$\beta = \min\limits_{x\in R}\{2^{6\sin 3x + 8\cos 3x}\}$

$= \min\limits_{n\in R}\{2^{[-10,10]}\}$

$\beta = 2^{-10}$

$\alpha^{1/5} = 2^2 = 4 \quad ,\beta^{1/5} = 2^{-2} = \frac{1}{4}$

$\because\ 8x^2 + bx + c = 0 \left\langle \begin{array}{l} \alpha^{1/5} = 4 \\ \beta^{1/5} = \frac{1}{4} \end{array} \right.$

using sum of roots: $\frac{-b}{8} = 4 + \frac{1}{4} \qquad \therefore b = -34$

using product of roots: $\frac{c}{8} = 4 \times \frac{1}{4} \qquad \therefore c = 8$

$\frac{c}{8} = 4 \times \frac{1}{4} \qquad \therefore c = 8$

$\therefore c - b = 42$

2. Let N be the set of natural numbers and a relation R on N be defined by

$R = \{(x, y) \in N \times N : x^3 - 3x^2y - xy^2 + 3y^3 = 0\}$. Then the relation R is :

(1) reflexive but neither symmetric nor transitive

(2) symmetric but neither reflexive nor transitive

(3) an equivalence relation

(4) reflexive and symmetric, but not transitive

Ans. Official Answer NTA (1)

Sol. $(x, y) \in N \times N$

$x^3 - 3x^2y - xy^2 + 3y^3 = 0$

$\Rightarrow$ $x(x^2 - y^2) - 3y(x^2 - y^2) = 0$

$\Rightarrow$ $(x - 3y)(x^2 - y^2) = 0$

$\Rightarrow$ $(x - y)(x + y)(x - 3y) = 0$

$\therefore$ $x = y$ or $x = -y$ or $x = 3y$

It is a reflexive relation Since xRx

$\because$ $x = 3y$

Consider $(3, 1) \in R$

But $(1, 3) \notin R$

$\because$ Hence is is not Symmetric

For transitive Consider $(9, 3) \in R$ and $(3, 1) \in R$

but $(9, 1)) \notin R$

$\therefore$ It is not transitive.

3. The value of $\lim\limits_{x\to 0}\left(\dfrac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right)$ is equal to :

(1) 4 (2) – 4 (3) – 1 (4) 0

Ans. Official Answer NTA (2)

Sol. $\lim\limits_{n\to o}\left(\dfrac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right)$

$\lim\limits_{n\to o}\left(\dfrac{x}{(1-\sin x)^{\frac{1}{8}}-(1+\sin x)^{\frac{1}{8}}}\right)$

$$=\lim_{n\to 0}\left(\frac{x}{\left(1-\frac{1}{8}\sin x+\frac{\frac{1}{8}\left(\frac{1}{8}-1\right)}{2!}\sin^2 x-\frac{\frac{1}{8}\left(\frac{1}{8}-1\right)\left(\frac{1}{8}-2\right)}{3!}\sin^3 x......\right)-\left(1+\frac{1}{8}\sin x+\frac{\frac{1}{8}\left(\frac{1}{8}-1\right)}{2!}\sin^2 x+\frac{\frac{1}{8}\left(\frac{1}{8}-1\right)\left(\frac{1}{8}-2\right)}{3!}\sin^3 x+.....\right)}\right)$$

$$=\lim_{n\to o}\frac{x}{\frac{-2}{8}\sin x-\frac{2\cdot\frac{1}{8}\left(\frac{1}{8}-1\right)\left(\frac{1}{8}-2\right)}{3!}\sin^3 x.....}$$

$$\lim_{n\to o}\frac{x}{\frac{-2}{8}\sin x\left[1+\frac{\left(\frac{1}{8}-1\right)\left(\frac{1}{8}-2\right)}{3!}\sin^2 x+.....\right]}$$

$$=\frac{1}{\frac{-2}{8}\cdot[1+0]}=-4$$

4. Let $\mathbb{C}$ be the set of all complex numbers. Let

$S_1=\{z\in\mathbb{C}:|z-2|\le 1\}$ and $S_2=\{z\in\mathbb{C}:z(1+i)+\overline{z}(1-i)\ge 4\}$.

Then, the maximum value of $\left|z-\frac{5}{2}\right|^2$ for $z\in S_1\cap S_2$ is equal to :

(1) $\frac{3+2\sqrt{2}}{4}$ (2) $\frac{5+2\sqrt{2}}{2}$ (3) $\frac{3+2\sqrt{2}}{2}$ (4) $\frac{5+2\sqrt{2}}{4}$

Ans. Official Answer NTA (4)

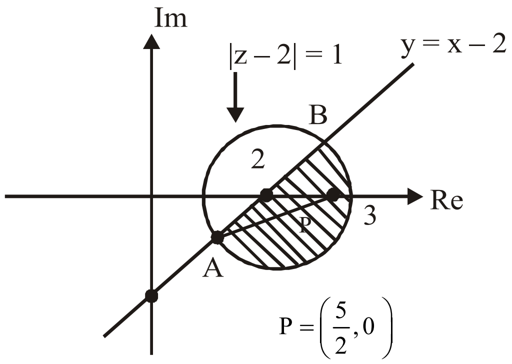
Sol. $S_1 : |z - 2| \leq|$ ....(i)

$S_2: z(1 + i) + \bar{z}(1 - i) \geq 4$

$(x + iy)(1 + i) + (x - iy)(1 - i) \geq 4$

$x - y \geq 2$

$y \leq x - 2$ ....(ii)



Equation of circle $(x - 2)^2 + y^2 = 1$ ....(iii)

for A & B $\quad y^2 + y^2 = 1$

$2y^2 = 1$

$y = \pm \dfrac{1}{\sqrt{2}}$

$A = \left(2 - \dfrac{1}{\sqrt{2}}, \dfrac{-1}{\sqrt{2}}\right)$

maximum value of $\left|z - \dfrac{5}{2}\right| = (PA)^2$

$= \left|2 - \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i - \dfrac{5}{2}\right|^2$

$= \left|\dfrac{-1}{2} - \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i\right|^2 = \dfrac{5 + \sqrt{2}}{4}$

5. If $\tan\left(\dfrac{\pi}{9}\right)$, x, $\tan\left(\dfrac{7\pi}{18}\right)$ are in arithmetic progression and $\tan\left(\dfrac{\pi}{9}\right)$, y, $\tan\left(\dfrac{5\pi}{18}\right)$ are also in arithmetic progression, then $|x - 2y|$ is equal to :

(1) 0 (2) 3 (3) 1 (4) 4

Ans. Official Answer NTA (1)

Sol. $2x = \tan\frac{\pi}{9} + \tan\left(\frac{7\pi}{18}\right)$

$= \tan\frac{\pi}{9} + \tan\left(\frac{\pi}{2} - \frac{\pi}{9}\right)$

$= \tan\frac{\pi}{9} + \cot\frac{\pi}{9}$

$x = \frac{1}{2}\left(\tan\frac{\pi}{9} + \cot\frac{\pi}{9}\right)$

$2y = \tan\frac{\pi}{9} + \tan\frac{5\pi}{18}$

$|x - 2y| = \left|\frac{1}{2}\left(\cot\frac{\pi}{9} - \tan\frac{\pi}{9}\right) - \tan\frac{5\pi}{18}\right|$

$= \left|\frac{1}{2}\left(2\cdot\cot\frac{2\pi}{9}\right) - \tan\frac{5\pi}{18}\right|$

$= \left|\cot\frac{2\pi}{9} - \tan\frac{5\pi}{18}\right|$

$= \left|\cot 40^\circ - \tan 50^\circ\right|$

$= \left|\tan 50^\circ - \tan 50^\circ\right| = 0$

6. Let A and B be two 3 × 3 real matrices such that $(A^2 - B^2)$ is invertible matrix. If $A^5 = B^5$ and $A^3B^2 = A^2B^3$, then the value of the determinant of the matrix $A^3 + B^3$ is equal to :

(1) 2 (2) 0 (3) 4 (4) 1

Ans. Official Answer NTA (2)

Answer by Matrix (Bonus)

Sol. $(A^3 + B^3)(A^2 - B^2) = A^5 - B^5 - A^3B^2 + B^3A^2$

Assuming $A^3B^2 = B^3A^2$ (even though it is not given in the question.)

$(A^3 + B^3)(A^2 - B^2) = 0$

Since $|A^2 - B^2| \neq 0 \Rightarrow |A^3 + B^3| = 0$

7. Let $f:[0,\infty)\rightarrow[0,3]$ be a function defined by $f(x)=\begin{cases}\max\{\sin t: 0\le t\le x\}, 0\le x\le\pi\\ 2+\cos x, \quad x>\pi\end{cases}$ Then which of the following is true ?

(1) f is not continuous exactly at two points in $(0,\infty)$

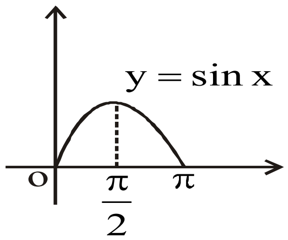
(2) f is differentiable everywhere in $(0,\infty)$

(3) f is continuous everywhere but not differentiable exactly at one point in $(0,\infty)$

(4) f is continuous everywhere but not differentiable exactly at two points in $(0,\infty)$

Ans. Official Answer NTA (2)

Sol.



$$f(x)=\begin{cases}\sin x & 0\le x<\frac{\pi}{2}\\ 1 & \frac{\pi}{2}\le x\le\pi\\ 2+\cos x & x>\pi\end{cases}$$

Doubtful points: $x=\frac{\pi}{2},\pi$

$\text{LHL}=\text{RHL}=f\left(\frac{\pi}{2}\right)=1$ $\therefore$ hence Continuous at $x=\frac{\pi}{2}$

$\text{LHL}=\text{RHL}=f(\pi)=1$ $\therefore$ hence Continuous at $x=\pi$

$$f'(x)=\begin{cases}\cos x & 0<x<\frac{\pi}{2}\\ 0 & \frac{\pi}{2}<x<\pi\\ -\sin x & x>\pi\end{cases}$$

$f'\left(\frac{\pi^-}{2}\right)=F'\left(\frac{\pi^+}{2}\right)=o$ $\therefore$ Derivable at $x=\frac{\pi}{2}$

$f'(\pi^-) = f'(\pi^+) = 0$ $\therefore$ Derivable at $x = \pi$

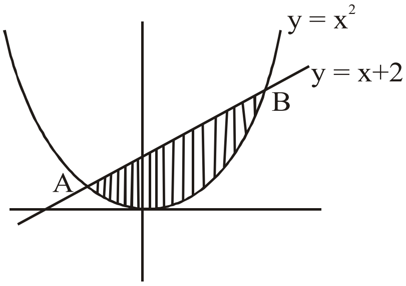
$\therefore$ f is differentiable everywhere in $(0, \infty)$

8. The area of the region bounded by $y - x = 2$ and $x^2 = y$ is equal to :

(1) $\frac{16}{3}$ (2) $\frac{4}{3}$ (3) $\frac{2}{3}$ (4) $\frac{9}{2}$

Ans. Official Answer NTA (4)

Sol.



For x co-ordinate of A and B

$x^2 = x+2$

$\Rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$\therefore x = -1, 2$

Required Area:

$$= \int_{-1}^{2} \left((x+2) - x^2\right) dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[2 + 4 - \frac{8}{3}\right] - \left[\frac{1}{2} - 2 + \frac{1}{3}\right]$$

$$= \frac{10}{3} - \left[\frac{3 - 12 + 2}{6}\right]$$

$$= \frac{10}{3} - \left[\frac{3 - 12 + 2}{6}\right]$$

$$= \frac{9}{2}$$

9. A student appeared in an examination consisting of 8 true - false type questions. The student guesses the answers with equal probability. The smallest value of n, so that the probability of guessing at least 'n' correct answers is less than $\frac{1}{2}$, is :

(1) 6 (2) 3 (3) 5 (4) 4

Ans. Official Answer NTA (3)

Sol. Let guessing Correct answer is Considered as Success.

$\therefore$ Probability of Success $p=\frac{1}{2}$

$q=\frac{1}{2}$

Now Prob. of at least n success

$$\sum_{r=n}^{8} {}^{8}C_r.\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{8-r}<\frac{1}{2}$$

$$\Rightarrow \quad \sum_{r=n}^{8} {}^{8}C_r.\frac{1}{2^8}<\frac{1}{2}$$

$$\sum_{r=n}^{8} {}^{8}C_r.<2^7$$

${}^8C_0 + {}^8C_1 + {}^8C_2 + \ldots\ldots + {}^8C_8 = 2^8$.

${}^8C_n + {}^8C_{n+1} + \ldots\ldots + {}^8C_8 < 2^7$.

$\Rightarrow$ ${}^8C_5 + {}^8C_6 + \ldots\ldots + {}^8C_8 < 2^7$.

$\therefore$ Smallest Value of n = 5

10. Let $\vec{a},\vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a}=\vec{b}\times(\vec{b}\times\vec{c})$. If magnitudes of the vectors $\vec{a},\vec{b}$ and $\vec{c}$ are $\sqrt{2},1$ and 2 respectively and the angle between $\vec{b}$ and $\vec{c}$ is $\theta\left(0<\theta<\frac{\pi}{2}\right)$, then the value of $1+\tan\theta$ is equal to :

(1) $\frac{\sqrt{3}+1}{\sqrt{3}}$ (2) $\sqrt{3}+1$ (3) 1 (4) 2

Ans. Official Answer NTA (4)

Sol. $\vec{a}=\vec{b}\times(\vec{b}\times\vec{c})$ .... (i)

$\vec{a}=(\vec{b}.\vec{c})\vec{b}-(\vec{b}.\vec{b})\vec{c}$

Take dot producet with $\vec{b}$ in (i)

$\because \vec{a}.\vec{b} = 0$ .... (ii)

$\vec{a} = (2\cos\theta\vec{b} - \vec{c}$ .... (iii)

Dot product with $\vec{a}$ in

$\vec{a}.\vec{b} = (2\cos\theta)\vec{b}.\vec{a} - \vec{c}.\vec{a}$

$\Rightarrow \quad 2 = 0 - \vec{c}.\vec{a}$

$\therefore \quad \vec{c}.\vec{a} = -2$ .... (iv)

Take dot product with $\vec{c}$ in (iii)

$\vec{a}.\vec{c} = (2\cos\theta)\vec{b}.\vec{c} - \vec{c}.\vec{c}$

$2 = 4\cos^2\theta$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\cos^2 \pm = \frac{1}{\sqrt{2}}$

$\theta \in \left(0, \frac{\pi}{2}\right)$

$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\theta = 45^o$

$1 + \tan\theta = 1 + \tan 45^o = 2$

11. Let the mean and variance of the frequency distribution

x :	$x_1 = 2$	$x_2 = 6$	$x_3 = 8$	$x_4 = 9$
f	4	4	α	β

be 6 and 6.8 respectively. If x_3 is changed from 8 to 7, then the mean for the new data will be:

(1) 5 (2) 4 (3) $\frac{16}{3}$ (4) $\frac{17}{3}$

Ans. Official Answer NTA (4)

Sol. $\mu = \frac{\sum f_i x_i}{\sum f_i}$

$\Rightarrow \mu = \frac{2\times4+6\times4+8\times\alpha+9\times\beta}{4+4+\alpha+\beta}$

$\Rightarrow 6 = \frac{32+8\alpha+9\beta}{8+\alpha+\beta}$

$\Rightarrow 48 + 6\alpha + 6\beta = 32 + 8\alpha + 9\beta$

$\Rightarrow 2\alpha + 8\beta = 16$ ....(i)

$$\sigma^2 = \frac{\sum f_i.(x_i - \mu)^2}{\sum f_i}$$

$$\Rightarrow 6.8 = \frac{4\times16 + 4\times0 + \alpha\times4 + \beta\times9}{8+\alpha+\beta}$$

$$\Rightarrow \frac{34}{5} = \frac{4 + 4\alpha + 9\beta}{8+\alpha+\beta}$$

$\Rightarrow 14\alpha - 11\beta = 48$ ....(ii)

On solving (i) & (ii)

$\alpha = 5$ and $\beta = 2$

New mean (μ) =

$$\frac{4\times2 + 4\times6 + \alpha\times7 + \beta\times9}{8+\alpha+\beta}$$

$$= \frac{8 + 24 + 5\times7 + 2\times9}{8+5+2}$$

$$= \frac{85}{15} = \frac{17}{3}$$

12. A possible value of 'x', for which the ninth term in the expansion of

$$\left\{3^{\log_3\sqrt{25^{x-1}+7}} + 3^{\left(-\frac{1}{8}\right)\log_3(5^{x-1}+1)}\right\}^{10}$$ in the increasing powers of $3^{\left(-\frac{1}{8}\right)\log_3(5^{x-1}+1)}$

is equal to 180, is :

(1) 1 (2) 0 (3) – 1 (4) 2

Ans. Official Answer NTA (1)

Sol. Required within term in the given expansion is

$${}^{10}C_8 . \left(3^{\log_3\left(\sqrt{25^{x-1}+7}\right)}\right)^2 . \left(3^{\left(-\frac{1}{8}\right)\log_3\left(5^{x-1}+1\right)}\right)^8 = 180$$

$$\Rightarrow \frac{10\times9}{2}.\left(\sqrt{25^{x-1}+7}\right)^2\left(3^{-\log_3\left(5^{x-1}+1\right)}\right) = 180$$

$$\Rightarrow 45\ (25^{x-1} + 7)\ .\left(3^{\log_3\left(\frac{1}{5^{x-1}+1}\right)}\right) = 180$$

$\Rightarrow 45\left(\left(5^{x-1}\right)^2+7\right).\left(\frac{1}{5^{x-1}+1}\right) = 180$

$\Rightarrow \frac{\left(\left(5^{x-1}\right)^7+7\right)}{5^{x-1}+1} = 4$

Put $5^{x-1} = t$

$\Rightarrow \frac{t^2+7}{t+1} = 4$

$\Rightarrow t^2 - 4t + 3 = 0$

$(t - 1)(t - 3) = 0$

$t = 1$ or $\quad t = 3$

$5^{x-1} = 0$ or $\quad 5^{x-1} = 3$

$x - 1 = 0 \quad x = (\log_5 3) + 1$

$x = 1$

13. For real numbers α and $\beta \neq 0$, if the point of intersection of the straight lines

$\frac{x-\alpha}{1} = \frac{y-1}{2} = \frac{z-1}{3}$ and $\frac{x-4}{\beta} = \frac{y-6}{3} = \frac{z-7}{3}$,

lies on the plane $x + 2y - z = 8$ then $\alpha - \beta$ is equal to:

(1) 5 (2) 7 (3) 3 (4) 9

Ans. Official Answer NTA (2)

Sol. Given lines :

$L_1 : \frac{x-\alpha}{1} = \frac{y-1}{2} = \frac{z-1}{3}$

$L_2 : \frac{x-4}{\beta} = \frac{y-6}{3} = \frac{z-7}{3}$

General point on the line L_1 :

$= (\alpha + t_1, 1 + 2t_1, 1 + 3t_1)$

General point on the line L_2 :

$= (4 + \beta + t_2, 6 + 3t_2, 7 + 3t_2)$

for point of intersection of 4 L_2

$\alpha + t_1 = 4 + \beta t_2$ ....(i)

$1 + 2t_2 = 6 + 3t_2$ ....(ii)

$1 + 3t_1 = 7 + 3t_2$ ....(iii)

On solving (ii) & (iii)

$t_1 = 1, t_2 = -1$

Using (i) $\alpha + 1 = 4 - \beta\ d$

$\alpha + \beta = 3$ ....(iv)

Point of intersection of L_1 & L_2 in term of α

$= (\alpha + 1, 1 + 2, 1 + 3)$

$= (1 + \alpha, 3, 4)$ lies on the plane $x + 2y - z = 8$

$1 + \alpha + 6 - 4 = 8 \qquad \alpha = 5, \beta = 2$

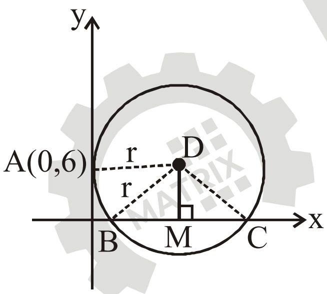
$\alpha - \beta = 7$

14. Consider a circle C which touches the y-axis at (0, 6) and cuts off an intercept $6\sqrt{5}$ on the x-axis. Then the radius of the circle C is equal to :

(1) 8 (2) $\sqrt{53}$ (3) 9 (4) $\sqrt{82}$

Ans. Official Answer NTA (3)

Sol.



$AD = BD = CD = r$ (radius)

$BC = 6\sqrt{5}$

$BM = CM = \frac{6\sqrt{5}}{2} \qquad [DM \perp^r BC]$

$= 3\sqrt{5}$

$DM = 6$

In Δ BDM

$BD^2 = BM^2 + DM^2$

$\Rightarrow r^2 = 45 + 36$

$r^2 = 81$

$r = 9$

15. The point P (a, b) undergoes the following three transformations successively :

(a) reflection about the line y = x.

(b) translation through 2 units along the positive direction of x-axis.

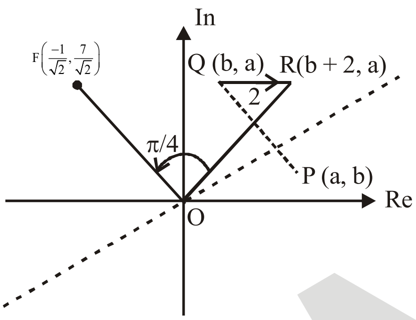
(c) rotation through angle $\frac{\pi}{4}$ about the origin in the anti-clockwise direction.

If the co-ordinates of the final position of the point P are $\left(-\frac{1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$, then the value of 2a + b is equal to :

(1) 9 (2) 13 (3) 7 (4) 5

Ans. Official Answer NTA (1)

Sol.



Using rotation theorem

$\Rightarrow \left(\frac{-1}{\sqrt{2}} + \frac{7}{\sqrt{2}} i\right) = (b+2+ai)e^{i\frac{\pi}{4}}$

$\Rightarrow (-1 + 7i) = (b + 2 + ai)(1 + i)$

Equating Real and imaginary parts

$\Rightarrow -1 = b + 2 - a \quad \therefore a - b = 3$ ....(i)

$\Rightarrow 7 = b + 2 + a \quad \therefore a + b = 5$ ....(ii)

Solving (i) & (ii)

a = 4, b = 1

2a + b = 8 + 1 = 9

16. Let $f : (a, b) \to R$ be twice differentiable function such that $f(x) = \int_a^x g(t)\,dt$ for a differentiable function g(x). If f (x) = 0 has exactly five distinct roots in (a, b), then g(x)g'(x)=0 has at least :

(1) seven roots in (a, b) (2) twelve roots in (a, b)

(3) five roots in (a, b) (4) three roots in (a, b)

Ans. Official Answer NTA (1)

Sol. $f'(x) = g(x)$

$f''(x) = g'(x)$

$g(x) \cdot g'(x) = 0$

$f'(x) \cdot f''(x) = 0$ ....(1)

Given $f(x) = 0$ has 5 roots in (a, b)

$\therefore$ If $f(x) = 0$ has at least 5 roots

$\Rightarrow$ $f'(x) = 0$ has atleast 4 roots

$\Rightarrow$ $f''(x) = 0$ has atleast 3 roots

$f'(x) \cdot f''(x) = 0$ has atleast 7 roots

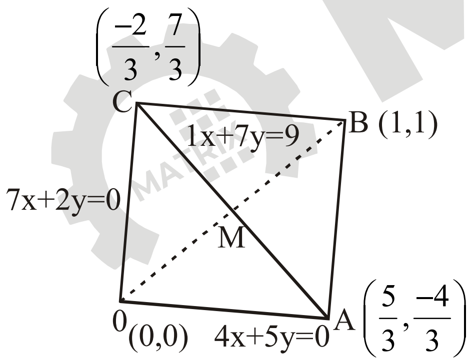
$g(x) \cdot g'(x) = 0$ has atleast 7 roots.

17. Two sides of a parallelogram are along the lines $4x + 5y = 0$ and $7x + 2y = 0$. If the equation of one of the diagonals of the parallelogram is $11x + 7y = 9$, then other diagonal passes through the point :

(1) (2, 2) (2) (1, 3) (3) (1, 2) (4) (2, 1)

Ans. Official Answer NTA (1)

Sol.



Mid-point of AC $= \left(\dfrac{\frac{5}{3} - \frac{2}{3}}{2}, \dfrac{\frac{-4}{3} + \frac{7}{3}}{2}\right)$

$= \left(\frac{1}{2}, \frac{1}{2}\right)$

Equation of OB (other diagonal) is :

$y - 0 = \frac{1-0}{1-0}(x-0)$

$y = x$

$\because (2, 2)$ lies on it

Ans: (2, 2)

18. Which of the following is the negation of the statement "for all $M > 0$, there exists $x \in S$ such that $x \geq M$" ?

(1) there exists $M > 0$, there exists $x \in S$ such that $x < M$

(2) there exists $M > 0$, such that $x < M$ for all $x \in S$

(3) there exists $M > 0$, there exists $x \in S$ such that $x \geq M$

(4) there exists $M > 0$, such that $x \geq M$for all $x \in S$

Ans. Official Answer NTA (2)

Sol. Negation of " for all " is there exists .

There exists $M > 0$, such that $x < M$ for all $x \in S$

19. Let $y = y(x)$ be the solution of the differential equation $(x - x^3)dy = (y + yx^2 - 3x^4)dx$, $x > 2$. If $y(3) = 3$, then $y(4)$ is equal to :

(1) 12 (2) 16 (3) 8 (4) 4

Ans. Official Answer NTA (1)

Sol. $(x - x^3)dy = (y + yx^2 - 3x^4)\, dx$

$\Rightarrow\ x\, dy - y\, dx = x^2\, (xdy + ydx) = 3x^4\, dx$

$\Rightarrow\ = \frac{xdy - ydx}{x^2}\ xdy + ydx - 3x^2dx$

$\Rightarrow\ \left(\frac{y}{x}\right) = d(xy) - d(x^3)$

Integrating both sides

$\frac{y}{x} = xy - x^3 + c$

$\therefore\ y(3) = 3$

$\frac{3}{3} = 3 \times 3 - 27 + c$

$c = 19$

$\therefore \ \frac{y}{x} \ xy - xy - x^3 + 19$ ....(1)

for y (4), substitute x = 4 in ....(1)

$\frac{y(4)}{4} = 4y\,(4) - 64 + 19$

$\Rightarrow \ y\,(4) = 16y\,(4) - 4 \times 45$

$15y\,(4) = 4 \times 55$

$y\,(4) = \frac{4 \times 45}{15} = 12$

20. Let $f : R \to R$ be defined as $f(x+y) + f(x-y) = 2f(x)f(y),\ f\left(\frac{1}{2}\right) = -1$. Then, the value of

$\sum_{k=1}^{20} \frac{1}{\sin(k)\sin(k+f(k))}$ is equal to :

(1) $\text{cosec}^2\,(21)\cos(20)\cos(2)$ (2) cosec2 (1) cosec2 (21) sin (20)

(3) $\sec^2\,(1)\sec(21)\cos(20)$ (4) sec2 (21) sin (20) sin (2)

Ans. Official Answer NTA (2)

Sol. $f(x + y) + f(x - y) = 2f(x).\ f(y)$ ....(1)

Put y = 0

$2f(x) = 2f(x).\ f(0)$

$\therefore 2f(x)(f(0) - 1) = 0$

$\because\ f\left(\frac{1}{2}\right) = -1, \qquad \therefore\ f(x) \neq 0$

$\therefore\ f(0) = 1$

$\therefore$ Put y = x

$f(2x) + f(0) = \left(f\left(\frac{1}{2}\right)\right)^2$

Put $x = \frac{1}{2}$

$f(1) + f(0) = \left(f\left(\frac{1}{2}\right)\right)^2$

$f(1) + 1 = 2 \times 1$

$f(1) = 1$

Put $y = 1$ in (1)

$f(x + 1) + f(x - 1) = 2f(x).\ f(1)$

$f(x + 1) + f(x - 1) = 2f(x).\ f(1)$

$f(x + 1) + f(x - 1) = 2f(x)$

$f(x + 1) - f(x) = f(x) - f(x - 1)$

Put $x = 1$

$f(2) - f(1) = f(1) - f(0)$

$x = 2$ $\quad f(3) - f(2) = f(2) - f(1)$

$x = 3$ $\quad f(4) - f(3) = f(3) - f(2)$

.

.

.

$x = n$ $\quad f(n+1) - f(n) = f(n) - f(n-1)$

$$\overline{f(n+1) - f(1) = f(n) - f(0)}$$

$\therefore f(0) = f(1) = 1$

$\Rightarrow f(n + 1) = f(n)$

$\therefore f(2) = f(3) = ... = f(20) = 1$

$\therefore f(k) = k = 1\ \forall\ k \in w$

$$\sum_{k=1}^{20} \frac{1}{\sin k \sin\left(k + f(k)\right)}$$

$$= \frac{1}{\sin 1} \sum_{k=1}^{20} \left[\cot k - \cot(k+1)\right]$$

$= \text{cosec}\ [\cot 1 - \cos 21]$

$$= \text{cosec}\ \frac{\sin\left(21 - 1\right)}{\sin 1 . \sin 21}$$

$= \text{cosec}^2\ 1.\ \sin 20.\ \text{cosec}\ 21$

SECTION - B

1. Let $A = \{n \in N\} |n^2 \leq + 10,000\}$, $B = \{3k + 1\} \mid k \in N\}$ and $C = \{2k |k \in N\}$, then the sum of all the elements of the set $A \cap (B - C)$ is equal to ________.

Ans. Official Answer NTA (832)

Sol. $A = \left\{n \in N \mid n^2 \leq n + 1000\right\}$

$B = \left\{3k + 1 \mid k \in N\right\}$

$C = \left\{2k \mid k \in N\right\}$

$\therefore$ For $n^2 \leq n + 10000$

$\Rightarrow n(n - 1) \leq 10000$

$\therefore$ Product of two consecutive nature No. ≤ 10000

$\therefore A = \{1, 2, 3, 4, ...,100\}$

$\therefore B = \{4, 7, 10 ......97, 100, 103......\}$

$C = \{2, 4, 6, 8, ...98, 100, 102......\}$

$\therefore B - C = \{7, 13, 19, ...91, 97\}$

$A \cap (B - C) = \{7, 13, 19 ......91, 97, \}$

No. or elements = 16

Sum of all the elements in $A \cap (B - C)$

$= 7 + 13 + 19 + ... + 91 + 97$

$= \frac{16}{2}\left[7 + 97\right] = 16 \times 52 = 832$

2. Let y = y(x) be the solution of the differential equation $dy = e^{\alpha x + y\ dx}$; $\alpha \in N$. If $y(\log_e 2) = \log_e 2$ and $y(0) = \log_e\left(\frac{1}{2}\right)$, then the value of α is equal to ________.

Ans. Official Answer NTA (2)

Sol. $dy = e^{\alpha x + y}\ dx$

$\Rightarrow e^{-y}\ dy = e^{\alpha x}\ dx$

Integrating both sides.

$-e^{-y} = \dfrac{e^{\alpha x}}{\alpha} + C$ ....(i)

$\therefore\ y(\ln 2) = \ln 2$

$\therefore\ -e^{-\ln 2} = \dfrac{e^{\alpha \ln 2}}{\alpha} + C$

$\Rightarrow -\dfrac{1}{2} = \dfrac{2^{\alpha}}{\alpha} + C$ ....(ii)

$\therefore\ y(0) = \ln\left(\dfrac{1}{2}\right)$

$\Rightarrow -e^{-\ln\left(\frac{1}{2}\right)} = \dfrac{1}{\alpha} + C$

$\Rightarrow -2 = \dfrac{1}{\alpha} + C$ ....(iii)

(ii)-(iii)

$\therefore\ \alpha \in N$

$\dfrac{3}{2} = \dfrac{2^{\alpha} - 1}{\alpha}$

$\alpha = 1,\ \dfrac{3}{2} = \dfrac{2-1}{1}$ (False)

$\alpha = 2,\ \dfrac{3}{2} = \dfrac{2^2-1}{1}$ (True)

$\therefore$ Value of $\alpha = 2$

3. If $\int_0^{\pi} (\sin^3 x) e^{-\sin^2 x}\, dx = \alpha - \dfrac{\beta}{e}\int_0^1 \sqrt{t}\, e^t dt$, then $\alpha + \beta$ is equal to _________.

Ans. Official Answer NTA (5)

Sol. $\int_0^{\pi} (\sin^3 x) e^{-\sin^2 x}\, dx = \alpha - \dfrac{\beta}{e}\int_0^1 \sqrt{t}\, e^t dt,$

$I = \int_0^{\pi} \sin^3 x\, e^{-\sin^2 x} dx$

$$I = 2\int_0^{\pi/2} \sin^3 x\, e^{-\sin^2 x} dx \quad \left[\begin{array}{l}\int_0^{2a} f(x)\,dx \\ = 2\int_0^{a} f(x)\,dx\end{array}\right. \qquad \because f(x) = f(2a - x)$$

$$I = 2\left[\int_0^{\pi/2} \left(1-\cos^2 x\right)\sin x\, e^{\cos^2 x - 1} dx\right]$$

Substitute $\cos^2 x = t$

$\Rightarrow -2\cos x \;\sin x\, dx = dt$

$$\Rightarrow \sin x\, dx = \frac{-dt}{2\cos x} = \frac{-dt}{2\sqrt{t}}$$

$$I = 2\left[\int_1^0 (1-t)e^{t-1}\left(\frac{-dt}{2\sqrt{t}}\right)\right]$$

$$= \int_0^1 \frac{(1-t)}{\sqrt{t}} e^{t-1}\; dt$$

$$= \frac{1}{e}\int_0^1 \frac{(1-t)}{\sqrt{t}} e^{t}\; dt$$

$$= \frac{1}{e}\left[\int_0^1 \frac{1}{\sqrt{t}} e^t + dt - \int_0^1 \sqrt{t}\; e^t dt\right]$$

$$= \frac{1}{e}\left[\left[2\sqrt{t}\, e^t\right]_0^1 - \int_0^1 2\sqrt{t}\, e^t\, dt - \int_0^1 \sqrt{t}\; e^t\; dt\right]$$

$$= \frac{1}{e}\left[2e - 3\int_0^1 \sqrt{t}\; e^t dt\right] = 2 - \frac{3}{e}\int_0^1 \sqrt{t}\; e^t dt$$

$\alpha = 2,\; \beta = 3$

4. Let n be a non-negative integer. Then the number of divisors of the form "4n + 1" of the number $(10)^{10}.(10)^{11}.(13)^{13}$ is equal to ________.

Ans. Official Answer NTA (924)

Sol. $N = (10)^{10}.\,(11)^{11}\,.\,(13)^{13}$

$= 2^{10}\,.5^{10}.\,(11)^{11}.\,(13)^{13}$

∴ To have divisior of the from '4n + 1'

These should not be any power of 2.

All divisors of 5^{10} and 13^{13} are of the form or '4n + 1'

So no. of divisors of $5^{10} \times 13^{13} = 11 \times 14$

Total number of divisors of 11^{11} is of the form (4n + 1) can be obtained iff we take even power of 11.

∴ So number of such divisors = 6 {0, 2, 4, 6, 8, 10}

∴ Total no. of divisors or the form '4n + 1' is

$= 11 \times 14 \times 6$

= 924.

5. The distance of the point P(3, 4, 4) from the point of intersection of the line joining the points Q(3, –4, –5) and R(2, –3, 1) and the plane 2x + y + z = 7, is equal to __________.

Ans. Official Answer NTA (7)

Sol. D.R. of the line QR

$= < -1, 1, 6 >$

General point on the line QR

$= (3 - t, -4 + t, -5 + 6t)$

Let It lies on the plane 2x + y + z = 7

$\Rightarrow 6 - 2t - 9 + 7t = 7$

$\Rightarrow 5t = 10$

$t = 2$

∴ Point of intersection of line QR and the plane = (3 – 2, – 4 + 2, –5 + 12)

= (1, –2, 7)

Required distance $= \sqrt{(3-1)^2 + (4+2)^2 + (4-7)^2}$

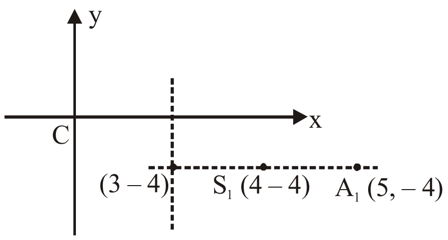
$= \sqrt{4+36+9}$

$= 7$

6. Let E be an ellipse whose axes are parallel to the co-ordinates axes, having its center at (3, –4), one focus at (4, –4) and one vertex at (5,–4). If $mx - y = 4$, $m > 0$ is a tangent to the ellipse E, then the value of $5m^2$ is equal to___________.

Ans. Official Answer NTA (3)

Sol.



$CA_1 = a = 2$

$CS_1 = ae = 1$

$\therefore\ e = \frac{1}{2}$

$b^2 = a^2 (1 - e^2)$

$= 4\left(1 - \frac{1}{4}\right) = 3$

Equation of the given ellipse is :

$$\frac{(x-3)^2}{4} + \frac{(y+4)^2}{3} = 1 \qquad ....(1)$$

$\therefore\ mx - y = 4$ is a tangent to (1)

$m(x - 3) + 3m = y + 4$

$\Rightarrow\ (y + 4) = m(x - 3) + 3m$

$\therefore$ this line is tangent to the above ellipse (1)

$\therefore$ Using condition of tangency

$$C^2 = a^2 m^2 + b^2$$
$$(3m)^2 = 4.\ m^2 + 3$$

$\Rightarrow 9m^2 = 4m^2 + 3$

$= 5m^2 = 3$

7. Let $\vec{a}=\hat{i}-a\hat{j}+\beta\hat{k}$, $\vec{b}=3\hat{i}+\beta\hat{j}-\alpha\hat{k}$ and $\vec{c}=-\alpha\hat{i}-2\hat{j}+\hat{k}$, where α and β are integers. If $\vec{a}\cdot\vec{b}=-1$ and $\vec{b}\cdot\vec{c}=10$, then $\left(\vec{a}\times\vec{b}\right).\vec{c}$ is equal to _________.

Ans. Official Answer NTA (9)

Sol. $\vec{a}=\hat{i}-\alpha\hat{j}+\beta\hat{k}$

$\vec{b}=3\hat{i}+\beta\hat{j}-\alpha\hat{k}$

$\vec{c}=-\alpha j-2\hat{j}+\hat{k}$

$\therefore\ \vec{a}.\vec{b}=1$

$\Rightarrow\ 3-\alpha\beta-\alpha\beta=-1$ $\therefore\ \alpha\beta=2$ ....(1)

$\therefore\ \vec{b}.\vec{c}=10$

$\therefore\ 3\alpha-2\beta-\alpha=10$

$4\alpha+2\beta=-10$

$\therefore\ 2\alpha+\beta=-5$ ....(2)

$\therefore\ \alpha,\beta\in$ integers and satisfying above conditions

$\therefore\ \alpha=-2$ and $\beta=-1$

$$\left(\vec{a}\times\vec{b}\right).\vec{c}=\begin{vmatrix}1 & -\alpha & \beta\\ 3 & \beta & -\alpha\\ -\alpha & -2 & 1\end{vmatrix}$$

$$=\begin{vmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 2 & -2 & 1\end{vmatrix}$$

$=1(-1+4)-2(3-4)-1(-6+2)$

$=3+2+4=9$

8. If the real part of the complex number $z = \frac{3+2i\cos\theta}{1-3i\cos\theta}.\theta \in \left(0, \frac{\pi}{2}\right)$ is zero, then the value of

$\sin^2 3\theta + \cos\theta$ is equal to__________.

Ans. Official Answer NTA (1)

Sol. $z = \frac{3+2i\cos\theta}{1-3i\cos\theta}$

$z = \frac{(3+2i+\cos\theta)(1+3i+\cos\theta)}{1+2\cos^2\theta}$

Real part of $z = \frac{3-6\cos^2\theta}{1+9\cos^2\theta} = 0$

$\therefore \cos^2\theta = \frac{1}{2}$ ....(1)

$\sin^2 3\theta + \cos^2\theta$

$= (3\sin^2\theta - 4\sin^3\theta)^2 + \cos^2\theta$

$= \sin^2\theta\,(3 - 4\sin^2\theta)^2 + \cos\theta$

$= \left(1-\frac{1}{2}\right)\left(4\times\frac{1}{2}-1\right)^2 + \frac{1}{2}$

$= \frac{1}{2}(2-1)^2 + \frac{1}{2}$

$= \frac{1}{2}+\frac{1}{2} = 1$

9. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $M = A + A^2 + A^3 + ... + A^{20}$, then the sum of all the elements of the matrix M is

equal to _________.

Ans. Official Answer NTA (2020)

Sol. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A = I + B (say)

$$B^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \therefore B^4 = B^5 = B^6 = ... = O$$

$M = A + A^2 + A^3 + ... + A^{20}$

$= (I + B) + (I + B)^2 + (I + B)^3 + ... + (I + B)^{20}$

$= 20\,I + (1 + {}^2C_1 + {}^3C_1 + ...{}^{20}C_1 +)\,B + (1 + {}^3C_2 + {}^4C_2 + ... + {}^{20}C_2)B^2$

$$= 20\,I + \left(\frac{20(20+1)}{2}\right)B + \left(\sum_{r=2}^{20}\frac{r(r-1)}{2}\right)B^2$$

$= 20\,I + 210\,B + 1330\,B^2$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 20 & 20 \end{bmatrix} + \begin{bmatrix} 0 & 210 & 210 \\ 0 & 0 & 210 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1330 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 210 & 1540 \\ 0 & 20 & 210 \\ 0 & 0 & 20 \end{bmatrix}$$

Sum of all the elements

$= 3 \times 20 + 2 \times 210 + 1540$

$= 2020$

10. The number of real roots of the equation $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1$is equal to____________.

Ans. Official Answer NTA (2)

Sol. $e^{4x} - e^{3x} - 4e^{2x} - e^x + 1 = 0$

$\Rightarrow \left(e^{4x} - e^{3x} + 1\right) - \left(e^{3x} + 2e^{2x} + e^x\right) = 0$

$\Rightarrow (e^{2x} - 1)^2 - e^x\,(e^x + 1)^2 = 0$

$\Rightarrow ((e^x)^2 -)^2 - e^x (e^x +1)^2 = 0$

$\Rightarrow \left(e^x+1\right)^2\left[\left(e^x-1\right)^2-e^x\right]=0$

$(e^x + 1)^2 > 0$

$\therefore (e^x -1)^2 - e^x = 0$

$(e^x)^2 -3e^x + 1 = 0$

Let ex = t

$t^2 = -3t + 1 = 0$

$t = \frac{3 \pm \sqrt{5}}{2}$

$e^x = \frac{3-\sqrt{5}}{2}$ or $e^x = \frac{3-\sqrt{5}}{2}$

$\therefore$ x $= \ln\left(\frac{3-\sqrt{5}}{2}\right)$ or x $= \ln\left(\frac{3-\sqrt{5}}{2}\right)$

$\therefore$ Number of real roots of the equation = 2