JEE Main February 2021

Question Paper With Text Solution

26 Feb.| Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911
Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE MAIN FEB 2021 | 26TH FEB SHIFT-2

SECTION – A

1. The sum of the series $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$ is equal to :

(1) $-\frac{41}{8}e+\frac{19}{8}e^{-1}-10$ (2) $\frac{41}{8}e+\frac{19}{8}e^{-1}+10$

(3) $\frac{41}{8}e-\frac{19}{8}e^{-1}-10$ (4) $\frac{41}{8}e+\frac{19}{8}e^{-1}-10$

Ans. Official Answer NTA : (3)

Sol. $\sum_{n=1}^{\infty}\frac{n^2+6n+10}{(2n+1)!}$

put 2n + 1 = r

for n = {1, 2, 3, ...}

r = {3, 5, 7, ...}

$$=\sum_{r=3,5,7,....}\frac{\left(\frac{r-1}{2}\right)^2+6\cdot\left(\frac{r-1}{2}\right)+10}{r!}$$

$$=\sum_{r=3,5,7,....}\frac{(r-1)^2+12(r-1)+10}{4\underline{|r}}$$

$$=\sum_{r=3,5,7,....}\frac{r^2+10r+29}{4\underline{|r}}$$

$$=\sum_{r=3,5,7,....}\frac{r(r-1)+11r+29}{4\underline{|r}}$$

$$=\sum_{r=3,5,7,....}\left(\frac{1}{4(r-2)!}+\frac{11}{4(r-1)!}+\frac{29}{4}\cdot\frac{1}{r!}\right)$$

$$=\frac{1}{4}\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+.......\right)+\frac{11}{4}\left(\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+........\right)+\frac{29}{4}\left(\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+........\right)$$

$$=\frac{1}{4}\left(\frac{e-\frac{1}{e}}{2}\right)+\frac{11}{4}\left(\frac{e+\frac{1}{e}-2}{2}\right)+\frac{29}{4}\left(\frac{e-\frac{1}{e}-2}{2}\right)$$

$$=\frac{1}{8}\left[e-\frac{1}{e}+11e+\frac{11}{e}-22+29e-\frac{29}{e}-58\right]$$

$$=\frac{1}{8}\left[41e-\frac{19}{e}-80\right]=\frac{41}{8}e-\frac{19}{8}e^{-1}-10$$

2. For x > 0, if $f(x)=\int_1^x \frac{\log_e t}{(1+t)}dt$, then $f(e)+f\left(\frac{1}{e}\right)$ is equal to :

(1) 1 (2) – 1 (3) 0 (4) $\frac{1}{2}$

Ans. Official Answer NTA : (4)

Sol. $f(x)=\int_1^x \frac{\ln t}{1+t}dt$ ....(i)

put $x=\frac{1}{x}$

$$f\left(\frac{1}{x}\right)=\int_1^{\frac{1}{x}} \frac{\ln(t)}{1+t}dt$$

put $t=\frac{1}{z}$

$$dt=\frac{-1}{z^2}dz$$

$$f\left(\frac{1}{x}\right)=\int_1^x \frac{-\ln z}{1+\frac{1}{z}}\left(-\frac{1}{z^2}\right)dz$$

$$f\left(\frac{1}{x}\right)=\int_1^x \frac{\ln z}{z(1+z)}dz$$

$f\left(\frac{1}{x}\right)=\int_1^x \frac{\ln t}{t(1+t)}dt$ ....(ii)

(i) + (ii)

$$f(x)+f\left(\frac{1}{x}\right)=\int_1^x \frac{\ln t}{t}dt$$

$$\Rightarrow f(x)+f\left(\frac{1}{x}\right)=\left[\frac{(\ln t)^2}{2}\right]_1^x$$

$\Rightarrow f(x)+f\left(\frac{1}{x}\right)=\frac{(\ln x)^2}{2}$

$\therefore f(e)+f\left(\frac{1}{e}\right)=\frac{1}{2}=\frac{1}{2}$

3. Let slope of the tangent line to a curve at any point P(x, y) be given by $\frac{xy^2+y}{x}$. If the curve intersects the line x + 2y = 4 at x = – 2, then the value of y, for which the point (3, y) lies on the curve, is :

(1) $-\frac{18}{19}$ (2) $\frac{18}{35}$ (3) $-\frac{18}{11}$ (4) $-\frac{4}{3}$

Ans. Official Answer NTA : (1)

Sol. $\frac{dy}{dx}=\frac{xy^2+y}{x}$

$\Rightarrow \frac{dy}{dx}=y^2+\frac{1}{x}y$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx}-\frac{1}{y}\cdot\frac{1}{x}=1$

put $-\frac{1}{y}=z$

$\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$

$\Rightarrow \frac{dz}{dx}+z\cdot\frac{1}{x}=1$

I.F = $e^{\int\frac{1}{x}dx}=x$

$\Rightarrow z\cdot x=\int x\cdot 1\,dx+c$

$\Rightarrow z\cdot x=\frac{x^2}{2}+c$

$\Rightarrow \therefore z=\frac{x}{2}+\frac{c}{x}$

$\therefore -\frac{1}{y}=\frac{x}{2}+\frac{c}{x}=\frac{x^2+2c}{2x}$

$\therefore y=\frac{-2x}{x^2+2c}$ ....(i)

$\because$ Curve intersect the line x + 2y = 4 at x = – 2

I.e. – 2 + 2y = 4 $\quad \therefore$ y = 3

$\therefore$ (– 2, 3) lies on the curve

$$\because 3 = \frac{-2\times(-2)}{4+2c}$$

$$\Rightarrow 12 + 6c = 4$$

$$6c = -8 \qquad \therefore c = -\frac{4}{3}$$

The equation of the required curve is

$$y = \frac{-2x}{x^2 - \frac{8}{3}}$$

$$\therefore y = \frac{6x}{8-3x^2}$$

when x = 3

$$y = \frac{18}{8-27} \qquad \therefore y = -\frac{18}{19}$$

4. Let f(x) be a differentiable function at x = a with f'(a) = 2 and f(a) = 4. Then $\lim\limits_{x\to a}\frac{xf(a)-af(x)}{x-a}$ equals :

(1) 2a – 4 (2) 2a + 4 (3) 4 – 2a (4) a + 4

Ans. Official Answer NTA : (3)

Sol. $\lim\limits_{x\to a}\frac{xf(a)-af(x)}{x-a} \quad \left(\frac{0}{0}\right)$

Apply L' Hospital Rule

$$= \lim_{x\to a}\frac{f(a)-a\cdot f^1(x)}{1}$$

$= f(a) - a\cdot f^1(a)$

= 4 – a·2

= 4 – 2a

5. Let F_1 (A, B, C) = $(A \wedge \sim B) \vee [\sim C \wedge (A \vee B)] \vee \sim A$ and F_2 (A, B) = $(A \vee B) \vee (B \to \sim A)$ be two logical expressions. Then :

(1) F_1 is not a tautology but F_2 is a tautology

(2) Both F_1 and F_2 are not tautologies

(3) F_1 and F_2 both are tautologies

(4) F_1 is a tautology but F_2 is not a tautology

Ans. Official Answer NTA : (1)

Sol.

A	B	C	$\sim B$	$A \wedge \sim B$	$A \vee B$	$\sim C$	$\sim C \wedge (A \vee B)$
T	T	T	F	F	T	F	F
T	F	F	T	T	T	T	T
T	T	F	F	F	T	T	T
T	F	T	T	T	T	F	F
F	T	T	F	F	T	F	F
F	F	F	T	F	F	T	F
F	T	F	F	F	T	T	T
F	F	T	T	F	F	F	F

$(A \wedge \sim B) \vee [-C \wedge (A \vee B)]$	$\sim A$	$(A \wedge \sim B) \vee [-C \wedge (A \wedge \sim B)] \vee \sim A$
F	F	F
T	F	T
T	F	T
T	F	T
F	T	T
F	T	T
T	T	T
F	T	T

F_1 is not a tautology

A	B	$\sim A$	$A \vee B$	$B \to \sim A$	$(A \vee B) \vee (B \to \sim A)$
T	F	F	T	T	T
T	T	F	T	F	T
F	T	T	T	T	T
F	F	T	F	T	T

F_2 is a tautology

6. Let A = {1, 2, 3, ..., 10} and $f : A \rightarrow A$ be defined as

$$f(k) = \begin{cases} k+1 & \text{if } k \text{ is odd} \\ k & \text{if } k \text{ is even} \end{cases}$$

Then the number of possible functions $g : A \rightarrow A$ such that gof = f is :

(1) ${}^{10}C_5$ (2) 10^5 (3) 5! (4) 5^5

Ans. Official Answer NTA : (2)

Sol. g(f(x)) = f(x)

$\Rightarrow$ g(x) = x, when x is even

so total number of functions g from A to A

$= 10^5 \cdot 1$

$= 10^5$

7. A natural number has prime factorization given by $n = 2^x 3^y 5^z$, where y and z are such that y + z = 5 and $y^{-1} + z^{-1} = \frac{5}{6}$, y > z. Then the number of odd divisors of n, including 1, is :

(1) 12 (2) 6x (3) 6 (4) 11

Ans. Official Answer NTA : (1)

Sol. $n = 2^x 3^y 5^z$

$\because$ y + z = 5 ....(i)

$\frac{1}{y} + \frac{1}{z} = \frac{5}{6}$ ....(ii)

$\frac{y+z}{yz} = \frac{5}{6}$ $\therefore$ yz = 6 ....(iii)

$\because$ y > z $\therefore$ y = 3, z = 2

$\therefore$ Number of odd divisors = 1·(3 + 1) (2 + 1) = 4 × 3 = 12

8. If $0 < a, b < 1$, and $\tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$, then the value of $(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + ...$ is :

(1) $\log_e\left(\frac{e}{2}\right)$ (2) $e^2 - 1$ (3) $\log_e 2$ (4) e

Ans. Official Answer NTA : (3)

Sol. $\because \tan^{-1} a + \tan^{-1} b = \frac{\pi}{4}$

$\Rightarrow \tan^{-1} a = \frac{\pi}{4} - \tan^{-1} b$

$\Rightarrow \tan\left(\tan^{-1} a\right) = \tan\left(\frac{\pi}{4} - \tan^{-1} b\right)$

$\Rightarrow a = \frac{1-b}{1+b}$

$\Rightarrow a + ab = 1 - b$

$a + b + ab = 1$ ....(i)

$\because (a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + ...$

$= \left(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + ...\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + ...\right)$

$= \ln(1 + a) + \ln(1 + b)$

$= \ln((1 + a)(1 + b))$

$= \ln(a + b + ab + 1)$

$= \ln(1 + 1) = \ln 2$

9. Consider the following system of equations :

$x + 2y - 3z = a$

$2x + 6y - 11z = b$

$x - 2y + 7z = c$,

where a, b and c are real constants. Then the system of equations :

(1) has no solution for all a, b and c
(2) has infinite number of solutions when $5a = 2b + c$
(3) has a unique solution when $5a = 2b + c$
(4) has a unique solution for all a, b and c

Ans. Official Answer NTA : (2)

Sol. $D = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{vmatrix}$

$= 1\,(42 - 22) - 2\,(14 + 11) - 3\,(-4 - 6)$

$= 20 - 50 + 30 = 0$

$D_x = \begin{vmatrix} a & 2 & -3 \\ b & 6 & -11 \\ c & -2 & 7 \end{vmatrix}$

$= a\,[42 - 22] - b\,[14 - 6] + c\,(-22 + 18)$

$= 20a - 8b - 4c$

$= 4\,[5a - 2b - c]$

$D_y = \begin{vmatrix} 1 & a & -3 \\ 2 & b & -11 \\ 1 & c & 7 \end{vmatrix}$

$= -5\,[5a - 2b - c]$

$D_z = \begin{vmatrix} 1 & 2 & a \\ 2 & 6 & b \\ 1 & -2 & c \end{vmatrix}$

$= -2\,[5a - 2b - c]$

for $5a - 2b - c = 0$, $D_x = D_y = D_z = 0$

$\therefore$ system has infinite number of solutions

when $5a = 2b + c$

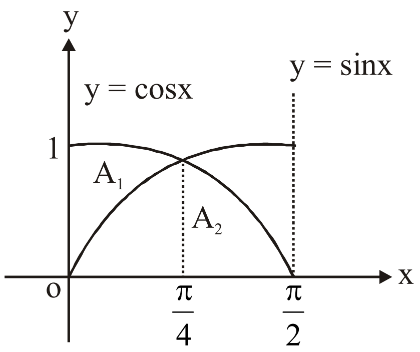
10. Let A_1 be the area of the region bounded by the curves y = sinx, y = cosx and y-axis in the first quadrant. Also, let A_2 be the area of the region bounded by the curves y = sinx, y = cosx, x-axis and $x = \frac{\pi}{2}$ in the first quadrant. Then,

(1) $A_1 = A_2$ and $A_1 + A_2 = \sqrt{2}$ (2) $2A_1 = A_2$ and $A_1 + A_2 = 1 + \sqrt{2}$

(3) $A_1 : A_2 = 1 : \sqrt{2}$ and $A_1 + A_2 = 1$ (4) $A_1 : A_2 = 1 : 2$ and $A_1 + A_2 = 1$

Ans. Official Answer NTA : (3)

Sol.



$$A_1 = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx = \left[\sin x + \cos x\right]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

$$A_2 = \int_0^{\frac{\pi}{4}} \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx$$

$$= \left[-\cos x\right]^{\frac{\pi}{4}} + \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}} = 2 - \sqrt{2}$$

$A_1 + A_2 = 1$, $\quad \frac{A_1}{A_2} = \frac{1}{\sqrt{2}}$

11. Let $f : R \to R$ be defined as :

$$f(x) = \begin{cases} 2\sin\left(-\frac{\pi x}{2}\right), & \text{if } x < -1 \\ |ax^2 + x + b|, & \text{if } -1 \le x \le 1 \\ \sin(\pi x), & \text{if } x > 1 \end{cases}$$

If f(x) is continuous on R, then a + b equals :

(1) 3 (2) 1 (3) – 3 (4) – 1

Ans. Official Answer NTA : (4)

Sol. Doubtful points : {– 1, 1}

at x = – 1 : $f(-1) = |a + b - 1|$

$LHL = 2\sin\frac{\pi}{2} = 2$

$RHL = |a + b - 1|$

for continuity at x = – 1

$|a + b - 1| = 2$ ....(i)

at x = 1 $f(1) = |a + b + 1|$

$LHL = |a + b + 1|$

$RHL = \sin\pi = 0$

for continuity at x = 1

$|a + b + 1| = 0$

$\Rightarrow a + b + 1 = 0$

$\therefore a + b = -1$ ....(ii)

$\because$ for a + b = – 1

$|a + b - 1| = |-1 - 1| = 2$

f(x) is continuous at x = – 1 and x = 1

$\Rightarrow$ f(x) is continuous on R

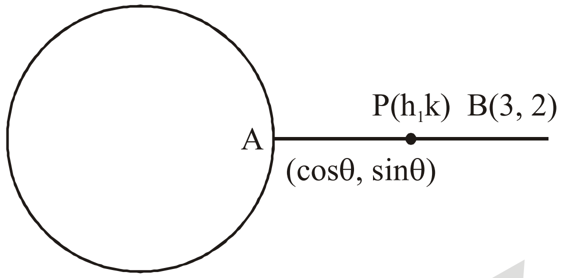
$\therefore a + b = -1$

12. If the locus of the mid-point of the line segment from the point (3, 2) to a point on the circle, $x^2 + y^2 = 1$ is a circle of radius r, then r is equal to :

(1) 1 (2) $\frac{1}{3}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

Ans. Official Answer NTA : (3)

Sol. P is the mid-point of AB



$h = \frac{3+\cos\theta}{2}$

$k = \frac{2+\sin\theta}{2}$

$\cos\theta = 2h - 3$

$\sin\theta = 2k - 2$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow (2k-2)^2 + (2h-3)^2 = 1$

$\left(h - \frac{3}{2}\right)^2 + (k-1)^2 = \frac{1}{4}$

$\therefore \left(x - \frac{3}{2}\right)^2 + (y-1)^2 = \left(\frac{1}{2}\right)^2$

$\therefore r = \frac{1}{2}$

13. If vectors $\overrightarrow{a_1} = x\hat{i} - \hat{j} + \hat{k}$ and $\overrightarrow{a_2} = \hat{i} + y\hat{j} + z\hat{k}$ are collinear, then a possible unit vector parallel to the vector $x\hat{i} + y\hat{j} + z\hat{k}$ is :

(1) $\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$ (2) $\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}-\hat{k}\right)$ (3) $\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}+\hat{k}\right)$ (4) $\frac{1}{\sqrt{2}}\left(\hat{i}-\hat{j}\right)$

Ans. Official Answer NTA : (3)

Sol. $\because \overrightarrow{a_1}$ and $\overrightarrow{a_2}$ are collinear

$\therefore \overrightarrow{a_1} = \lambda\overrightarrow{a_2}$

$\left(x\hat{i} - \hat{j} + \hat{k}\right) = \lambda\left(\hat{i} + y\hat{j} + z\hat{k}\right)$

$\therefore x = \lambda$

$y\lambda = -1$

$z\lambda = 1$

$\therefore y = -\frac{1}{\lambda}$

$\therefore z = \frac{1}{\lambda}$

$\therefore x\hat{i} + y\hat{j} + z\hat{k}$

$= \lambda\hat{i} - \frac{1}{\lambda}\hat{j} + \frac{1}{\lambda}\hat{k}$

$= \frac{1}{\lambda}\left(\lambda^2\hat{i} - \hat{j} + \hat{k}\right)$

$\therefore$ for $\lambda = 1$

a vector parallel to $x\hat{i} + y\hat{j} + z\hat{k}$

$= \hat{i} - \hat{j} + \hat{k}$

$\therefore$ unit vector = $\frac{1}{\sqrt{3}}\left(\hat{i} - \hat{j} + \hat{k}\right)$

14. Let A(1, 4) and B(1, – 5) be two points. Let P be a point on the circle $(x - 1)^2 + (y - 1)^2 = 1$ such that $(PA)^2 + (PB)^2$ have maximum value, then the points P, A and B lie on :

(1) a hyperbola (2) a straight line (3) a parabola (4) an ellipse

Ans. Official Answer NTA : (2)

Sol. General point on the circle P

$= (1 + \cos\theta, 1 + \sin\theta)$

$PA^2 + (PB)^2 = (\cos\theta)^2 + (\sin\theta - 3)^2 + (\cos\theta)^2 + (\sin\theta + 6)^2$

$= 2(\cos^2\theta + \sin^2\theta) - 6\sin\theta + 12\sin\theta$

$PA^2 + PB^2 = 2 + 12\sin\theta$

$PA^2 + PB^2$ have maximum value

when $\sin\theta = 1$

$\therefore \cos\theta = 0$

Required

$\therefore$ point P = (1, 2)

$\therefore$ Since x-coordinates of all the three points P, A and B are equal

$\therefore$ P, A and B lie on the straight line x = 1.

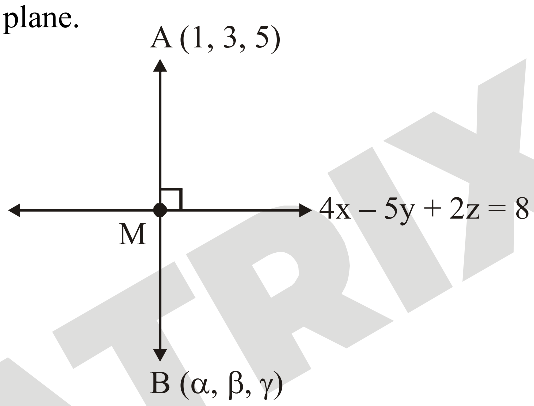
15. If the mirror image of the point (1, 3, 5) with respect to the plane $4x - 5y + 2z = 8$ is (α, β, γ), then $5(\alpha + \beta + \gamma)$ equals :

(1) 41 (2) 43 (3) 47 (4) 39

Ans. Official Answer NTA : (3)

Sol. B is the mirror image of A w.r.t. the plane $4x - 5y + 22 = 8$

M is the mid-point of AB, which lies on the plane.



DR of AB = DR of the normal

$= < 4, -5, 2 >$

Eq^n of the line AB :

$$\frac{x-1}{4} = \frac{y-3}{-5} = \frac{z-5}{2}$$

General point on AB

$= (1 + 4\lambda, 3 - 5\lambda, 5 + 2\lambda)$ (Say M)

For M :

$\because$ M lies on the plane

$4(1 + 4\lambda) - 5(3 - 5\lambda) + 2(5 + 2\lambda) = 8$

$\Rightarrow -1 + 45\lambda = 8$

$45\lambda = 9$

$\therefore \lambda = \frac{1}{5}$

$$M = \left(1 + \frac{4}{5}, 3 - \frac{5}{5}, 5 + \frac{2}{5}\right)$$

$$= \left(\frac{9}{5}, \frac{10}{5}, \frac{27}{5}\right)$$

$\because \frac{\alpha + 1}{2} = \frac{9}{5}$ $\therefore \alpha = \frac{13}{5}$

$\frac{\beta + 3}{2} = \frac{10}{5}$ $\therefore \beta = \frac{5}{5}$

$\frac{\gamma+5}{2}=\frac{27}{5}$ $\quad\therefore \gamma=\frac{29}{5}$

$\alpha+\beta+\gamma=\frac{47}{5}$

$\Rightarrow 5(\alpha+\beta+\gamma)=47$

16. A seven digit number is formed using digits 3, 3, 4, 4, 4, 5, 5. The probability, that number so formed is divisible by 2, is :

(1) $\frac{6}{7}$ (2) $\frac{3}{7}$ (3) $\frac{4}{7}$ (4) $\frac{1}{7}$

Ans. Official Answer NTA : (2)

Sol. Total number formed using the given digits = $\frac{\lfloor 7}{\lfloor 2\lfloor 3\lfloor 2}$

For the number be divisible by 2

$(--------)\underline{4}$

$=\frac{\lfloor 6}{\lfloor 2\lfloor 2\lfloor 2}$

$\therefore$ Required probability $=\frac{\frac{\lfloor 6}{\lfloor 2\lfloor 2\lfloor 2}}{\frac{\lfloor 7}{\lfloor 2\lfloor 3\lfloor 2}}$

$=\frac{\lfloor 6}{\lfloor 7}\times\frac{\lfloor 3}{\lfloor 2}=\frac{3}{7}$

17. Let f(x) = $\sin^{-1} x$ and $g(x)=\frac{x^2-x-2}{2x^2-x-6}$. If $g(2)=\lim_{x\to 2} g(x)$, then the domain of the function fog is :

(1) $(-\infty,-1]\cup[2,\infty)$ (2) $(-\infty,-2]\cup[-1,\infty)$

(3) $(-\infty,-2]\cup\left[-\frac{4}{3},\infty\right)$ (4) $(-\infty,-2]\cup\left[-\frac{3}{2},\infty\right)$

Ans. Official Answer NTA : (3)

Sol. $g(2) = \lim_{x\to 2}\frac{x^2-x-2}{2x^2-x-6}$

$= \lim_{x\to 2}\frac{(x-2)(x+1)}{(x-2)(2x+3)}$

$= \lim_{x\to 2}\frac{(x+1)}{2x+3} = \frac{3}{7}$

$$g(x) = \begin{cases} \frac{x+1}{2x+3} & x \neq 2 \\ \frac{3}{7} & x = 2 \end{cases}$$

$fog(x) = \sin^{-1}(g(x))$

for fog be defined

$-1 \le g(x) \le 1$

$\because$ for x = 2 $\quad g(2) = \frac{3}{7}$

$\therefore$ fog is defined when x = 2 ....(i)

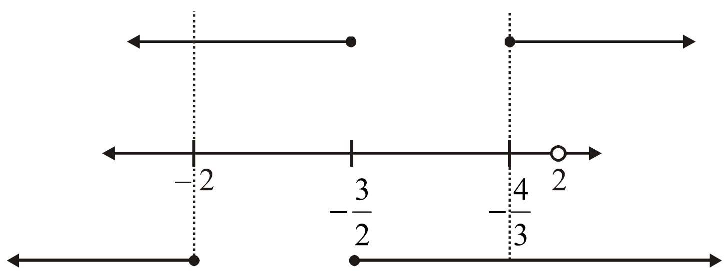
when $x \neq 2$

$-1 \le g(x) \le 1$

$-1 \le \frac{x+1}{2x+3} \le 1$

$\frac{x+1}{2x+3} \ge -1$ and $\frac{x+1}{2x+3} \le 1$

$\Rightarrow \frac{3x+4}{2x+3} \ge 0$ and $\frac{x+2}{2x+3} \ge 0$



$\therefore x \in (-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right) - \{2\}$ ....(ii)

Using (i) and (ii)

Domain of fog is :

$(-\infty, -2] \cup \left[-\frac{4}{3}, \infty\right)$

18. Let L be a line obtained from the intersection of two planes x + 2y + z = 6 and y + 2z = 4. If point P(α , β , γ) is the foot of perpendicular from (3, 2, 1) on L, then the value of 21($\alpha + \beta + \gamma$) equals :

(1) 102 (2) 142 (3) 68 (4) 136

Ans. Official Answer NTA : (1)

Sol. Let DR's of LOI are <a, b, c>

a + 2b + c = 0

0a + b + 2c = 0

$\frac{a}{4-1} = \frac{b}{0-2} = \frac{c}{1-0}$

$\because \frac{a}{3} = \frac{b}{-2} = \frac{c}{1}$

$\therefore$ DR' of line of intersection (LOI) = <3, –2, 1>

For a point on the LOI :

Let y = 0

x + z = 6 ....(i)

2z = 4 ....(ii)

on solving (i) and (ii)

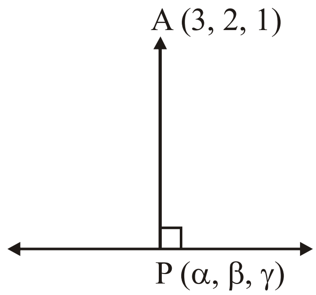
$\therefore$ x = 4, z = 2

$\therefore$ (4, 0, 2) lies on the LOI

General point on the line :

(x, y, z) = (4 + 3 λ , 0 – 2 λ , 2 + λ)

Let $(\alpha, \beta, \gamma) = (4 + 3\lambda, -2\lambda, 2 + \lambda)$



$\therefore \alpha = 4 + 3\lambda$

$\beta = -2\lambda$

$\gamma = 2 + \lambda$

DR of AP = $\langle 3\lambda + 1, -2\lambda - 2, \lambda + 1 \rangle$

$\therefore$ AP is $\perp^r$ to the line

$\therefore 3(3\lambda + 1) - 2(-2\lambda - 2) + 1(\lambda + 1) = 0$

$\Rightarrow 14\lambda + 8 = 0$

$\lambda = -\frac{4}{7}$

$\alpha + \beta + \gamma = 6 + 2\lambda$

$= 6 + 2\left(-\frac{4}{7}\right)$

$= \frac{42 - 8}{7} = \frac{34}{7}$

$21(\alpha + \beta + \gamma) = 21 \times \frac{34}{7} = 102$

19. Let $f(x) = \int_0^x e^t f(t)dt + e^x$ be a differentiable function for all $x \in R$. Then f(x) equals :

(1) $e^{(e^x - 1)}$ (2) $2e^{(e^x - 1)} - 1$ (3) $e^{e^x} - 1$ (4) $2e^{e^x} - 1$

Ans. Official Answer NTA : (2)

Sol. $f(x) = \int_0^x e^t f(t)\,dt + e^x$ ....(i)

Differentiate (i) w.r.t. 'x'

$f'(x) = e^x f(x) + e^x$

$\Rightarrow f'(x) = e^x (f(x) + 1)$

$\Rightarrow \frac{f'(x)}{f(x) + 1} = e^x$

Integrating both sides

$\int \frac{f'(x)}{f(x)+1}dx = \int e^x dx$

$\Rightarrow \ln(f(x)+1) = e^x + c$

$f(x) + 1 = e^{(e^x+c)}$

$f(x) = e^{e^x} \cdot e^c - 1$ ....(ii)

Put x = 0 in (i)

$f(0) = 0 + e^\circ = 1$

Put x = 0 in (ii)

$f(0) = e \cdot e^c - 1$

$\Rightarrow 1 = e \cdot e^c - 1$

$\Rightarrow e \cdot e^c = 2$

$\because e^c = \frac{2}{e}$

$f(x) = e^{e^x} \cdot \frac{2}{e} - 1$

$= 2\left(e^{(e^x-1)}\right) - 1$

20. The triangle of maximum area that can be inscribed in a given circle of radius 'r' is :

(1) An equilateral triangle of height $\frac{2r}{3}$.

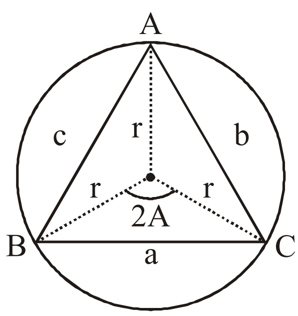
(2) A right angle triangle having two of its sides of length 2r and r.

(3) An equilateral triangle having each of its side of length $\sqrt{3}$ r.

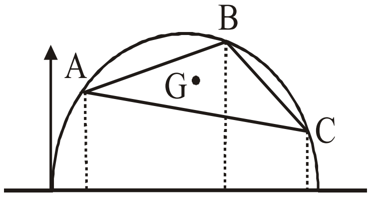
(4) An isosceles triangle with base equal to 2r.

Ans. Official Answer NTA : (3)

Sol. Area of Δ ABC



$\Delta = \dfrac{abc}{4r}$



Centroid $G = \left(\dfrac{A+B+C}{3}, \dfrac{\sin A + \sin B + \sin C}{3}\right)$

$\dfrac{\sin A + \sin B + \sin C}{3} \le \sin\left(\dfrac{A+B+C}{3}\right)$

$\sin A + \sin B + \sin C \le \dfrac{3\sqrt{3}}{2}$

$\dfrac{a}{2r} + \dfrac{b}{2r} + \dfrac{c}{2r} \le \dfrac{3\sqrt{3}}{2}$

$\Rightarrow \dfrac{a+b+c}{3} \le \sqrt{3}\, r$

$G.M \le A.M$

$(abc)^{\frac{1}{3}} \le \dfrac{a+b+c}{3}$

$\Rightarrow abc \le \left(\dfrac{a+b+c}{3}\right)^3$

$\Rightarrow \dfrac{abc}{4r} \le \dfrac{1}{4r}\left(\dfrac{a+b+c}{3}\right)^3$

$\Rightarrow \Delta \le \dfrac{1}{4r}\left(\dfrac{a+b+c}{3}\right)^3 \le \dfrac{1}{4r}\left(\sqrt{3}\, r\right)^3$

$\Delta \le \dfrac{3\sqrt{3}}{4} r^2$

$\Delta_{max} = \dfrac{3\sqrt{3}}{4} r^2$ (Equality holds when a = b = c)

Δ is an equilateral Δ

$a = \sqrt{3}\, r$

SECTION – B

1. The total number of 4-digit numbers whose greatest common divisor with 18 is 3, is __________.

Ans. Official Answer NTA : (1000)

Sol. GCD (N, 18) = 3

Number must be an odd number and multiple of 3 and not a multiple of 9.

4-digit odd numbers multiple of 3 are :

1005, 1011, ..., 9999

Total such numbers are = 1500

4-digit odd number multiple of 9 are :

1017, 1035, ..., 9999

Total such numbers = 500

Required numbers = 1000.

2. Let the normals at all the points on a given curve pass through a fixed point (a, b). If the curve passes through (3, – 3) and $\left(4,-2\sqrt{2}\right)$, and given that $a-2\sqrt{2}\ b=3$, then $(a^2 + b^2 + ab)$ is equal to _______.

Ans. Official Answer NTA : (9)

Sol. Equation of normal at P(x, y)

$$Y-y=\frac{1}{\frac{dy}{dx}}(X-x)$$

It passes through (a, b)

$$\Rightarrow b-y=-\frac{dx}{dy}\ (a-x)$$

$$\Rightarrow (b-y)\ dy=(x-a)\ dx$$

Integrate both sides

$$\int(b-y)\,dy=\int(x-a)\,dx$$

$$\Rightarrow by-\frac{y^2}{2}=\frac{x^2}{2}-ax$$

$x^2 + y^2 - 2ax - 2by = 0$

passes through (3, –3) and $(4, -2\sqrt{2})$

$9 + 9 - 6a + 6b = 0$

$\therefore\ a - b = 3$ ....(i)

$16 + 8 - 8a + 4\sqrt{2}\ b = 0$

$2a - \sqrt{2}\,b = 6$ ....(ii)

(i) × (2) – (ii)

$b(\sqrt{2} - 2) = 0$

$b = 0$

$\therefore\ a = 3$

$\therefore\ a^2 + b^2 + ab = 3^2 + 0 + 0$

= 9.

3. Let $X_1, X_2, \ldots\ldots\ldots X_{18}$ be eighteen observations such that $\sum_{i=1}^{18}(X_i - \alpha) = 36$ and $\sum_{i=1}^{18}(X_i - \beta)^2 = 90$, where α and β are distinct real numbers. If the standard deviation of these observations is 1, then the value of $|\alpha - \beta|$ is ________.

Ans. Official Answer NTA : (4)

Sol. $\sum_{i=1}^{18}(x_i - \alpha) = 36$

$\therefore\ \sum_{i=1}^{18} x_i - 18\alpha = 36$

$\sum_{i=1}^{18} x_i = 36 + 18\alpha$

$\sum_{i=1}^{18}(x_i - \beta)^2 = 90$

$\sum_{i=1}^{18} x_i^2 - 2\beta\sum_{i=1}^{18} x_i + 18\beta^2 = 90$

$\sum_{i=1}^{18} x_i^2 = 90 + 2\beta(36 + 18\alpha) - 18\beta^2$

Variance of these observations

$$6^2 = \frac{1}{18}\sum_{i=1}^{18} x_i^{\,2} - \left(\frac{\sum x_i}{18}\right)^2$$

$\Rightarrow 1 = \frac{1}{18}\ (90 + 2\beta\ (36 + 18\alpha) - 18\beta^2) - (2 + \alpha)^2$

$\Rightarrow 1 = 5 + 2\beta\ (2 + \alpha) - \beta^2 - (\alpha + 2)^2$

$\Rightarrow (\alpha + 2)^2 + \beta^2 - 2\beta(\alpha + 2) = 4$

$\Rightarrow (\alpha + 2 - \beta)^2 = 4$

$\Rightarrow (\alpha - \beta + 2)^2 - 4 = 0$

$\Rightarrow (\alpha - \beta + 4)\ (\alpha - \beta) = 0$

$\because\ \alpha \neq \beta$

$\therefore\ \alpha - \beta + 4 = 0$

$\alpha - \beta = -4$

$\therefore\ |\alpha - \beta| = 4$

4. Let a be an integer such that all the real roots of the polynomial $2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$ lie in the interval $(a, a + 1)$. Then, $|a|$ is equal to _________.

Ans. Official Answer NTA : (2)

Sol. Let $P(x) = 2x^5 + 5x^4 + 10x^3 + 10x^2 + 10x + 10$

$P'(x) = 10x^4 + 20x^3 + 30x^2 + 20x + 10$

$P'(x) = 10x^2\left[x^2 + 2x + 3 + \frac{2}{x} + \frac{1}{x^2}\right]$

$= 10x^2\left[x^2 + \frac{1}{x^2} + 2\left(x + \frac{1}{x}\right) + 1\right]$

$= 10x^2\left[\left(x + \frac{1}{x}\right)^2 + 2\left(x + \frac{1}{x}\right) + 1\right]$

$= 10x^2\left[x + \frac{1}{x} + 1\right]^2$

$P'(x) = 10\ [x^2 + x + 1]^2 > 0$

P(x) is an odd degree increasing polynomial function

$\Rightarrow$ only one real root

$P(0) = 10$

$P(-1) = -2 + 5 - 10 + 10 - 10 + 10 = 3$

$P(-2) = -64 + 80 - 80 + 40 - 20 + 10 = -34$

$P(-1)\,P(-2) < 0$

Using Intermediate value theorem

$\Rightarrow$ P(x) = 0 has at least on root in the (–2, –1)

Since P(x) = 0 has exactly one real root

$\therefore$ all real roots lie in the interval (–2, –1)

$\therefore$ a = –2 $\qquad$ |a| = 2.

5. Let L be a common tangent line to the curves $4x^2 + 9y^2 = 36$ and $(2x)^2 + (2y)^2 = 31$. Then the square of the slope of the line L is ________.

Ans. Official Answer NTA : (3)

Sol. $C_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ ....(i)

$C_2 : x^2 + y^2 = \frac{31}{4}$ ....(ii)

Let slope of common tangent be m.

Equetion of tangent of slope m to the ellipse (i)

$y = mx \pm \sqrt{9m^2 + 4}$ ....(iii)

$\therefore mx - y \pm \sqrt{9m^2 + 4} = 0$

this line is also tangent to circle (ii)

$$\frac{\left|\pm\sqrt{9m^2+4}\right|}{\sqrt{m^2+1}} = \sqrt{\frac{31}{4}}$$ [condition of tangency for the circle]

$$\Rightarrow \sqrt{9m^2+4} = \sqrt{\frac{31}{4}} \cdot \sqrt{m^2+1}$$

$\Rightarrow 9m^2 + 4 = \frac{31}{4}(m^2 + 1)$

$\Rightarrow 36\,m^2 + 16 = 31m^2 + 31$

$5m^2 = 15$

$m^2 = 3$

6. If the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$ satisfies the equation $A^{20} + \alpha A^{19} + \beta A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ for some real numbers α and β, then $\beta - \alpha$ is equal to ________.

Ans. Official Answer NTA : (4)

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^3 & 0 \\ 3 & 0 & -1 \end{bmatrix}$

$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By observing the pattren

$$A^{20} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{20} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{19} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{19} & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

$$A^{20} + 2A^{19} + \beta A = \begin{bmatrix} 1+\alpha+\beta & 0 & 0 \\ 0 & 2^{20}+\alpha\cdot 2^{19}+2\beta & 0 \\ 3\alpha+3\beta & 0 & 1-\alpha-\beta \end{bmatrix}$$

$\alpha + \beta = 0$ and $2^{20} + \alpha \cdot 2^{19} + 2\beta = 4$

$2^{20} + \alpha \cdot 2^{19} - 2\alpha = 4$

$\Rightarrow 2^{19} (\alpha + 2) = 2(\alpha + 2)$

$(\alpha + 2) (2^{19} - 2) = 0$

$\alpha = -2$

$\therefore\ \beta = 2$

$\therefore\ |\alpha - \beta| = |-2 -2| = |4|$

$\beta - \alpha = 2 - (-2) = 4.$

7. Let z be those complex numbers which satisfy $|z + 5| \le 4$ and $z(1 + i) + \bar{z}(1 - i) \ge -10$, $i = \sqrt{-1}$. If the maximum value of $|z + 1|^2$ is $\alpha + \beta\sqrt{2}$, then the value of $(\alpha + \beta)$ is _________.

Ans. Official Answer NTA : (48)

Sol. $|z + 5| \le 4$

$(x + 5)^2 + y^2 \le 16$ ....(i)

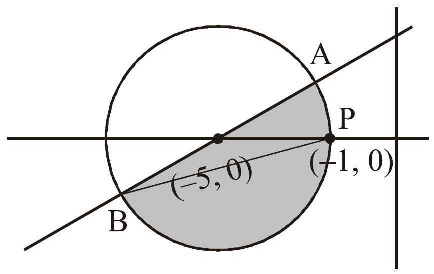
$z(1 + i) + \bar{z}(1 - i) \ge -10$

$(z + \bar{z}) + i(z - \bar{z}) \ge -10$

$\Rightarrow 2x + i\,(2i\,y) \ge -10$

$x - y \ge -5$ ....(ii)

$|z + 1|$ = Distance of z from $(-1, 0)$



$|z + 1|_{max} = PB$

for B

solve $(x + 5)^2 + y^2 = 16$ and $x - y = -5$

$x + 5 = y$

$y^2 + y^2 = 16$

$\therefore\ y = \pm\ 2\sqrt{2}$

for B, $y = -\ 2\sqrt{2}$

$\left(-5-2\sqrt{2},-2\sqrt{2}\right)$

$PB = \sqrt{\left(-4-2\sqrt{2}\right)^2+8}$

$|z+1|^2_{max} = PB^2 = 8+\left(4+2\sqrt{2}\right)^2$

$= 32+16\sqrt{2}$

$\alpha = 32,\ \beta = 16$

$\therefore\ \alpha + \beta = 48$

8. Let α and β be two real numbers such that $\alpha + \beta = 1$ and $\alpha\beta = -1$. Let $p_n = (\alpha)^n + (\beta)^n$, $p_{n-1} = 11$ and $p_{n+1} = 29$ for some integer $n \geq 1$. Then, the value of p_n^2 is ________.

Ans. Official Answer NTA : (324)

Sol. $\alpha + \beta = 1$

$\alpha\beta = -1$

$P_n = \alpha^n + \beta^n$

$P_{n+1} = a^{n+1} + b^{n+1}$

$= \alpha^n \cdot \alpha + \beta^n \cdot (\beta)$

$= \alpha^n \cdot (1 - \beta) + \beta^n \cdot (\beta)$

$= - \beta \cdot \alpha^n + \alpha^n + \beta^n - \alpha\beta^n$

$= - \alpha\beta (\alpha^{n+} + \beta^{n+}) (\alpha^n + \beta^n)$

$P_{n+1} = P_{n-1} + P_n$

for some $n \geq 1$

$29 = 11 + P_n$

$P_n = 18$

$P_n^2 = 324$

9. If $I_{m,n} = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, for $m, n \geq 1$, and $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}dx = \alpha \, Im, n, \alpha \in R$, then α equals _______.

Ans. Official Answer NTA : (1)

Sol. $I_{m,n} = \int_0^1 x^{m-1}\left(1-x\right)^{n-1} dx$

put $x = \frac{1}{1+t}$ $\quad \therefore t = \frac{1}{x} - 1$

$dx = \frac{-1}{(1+t)^2} dt$

$= \int_\infty^0 \frac{1}{(1+t)^{m-1}}\left(1 - \frac{1}{1+t}\right)^{n-1}\left(\frac{-1}{(1+t)^2}\right)dt$

$= -\int_\infty^0 \frac{t^{n-1}}{(1+t)^{m-1}(1+t)^{n-1}}\left(\frac{1}{(1+t)^2}\right)dt$

$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m-1}(1+t)^{n-1}(1+t)^2} dt$

$= \int_0^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt$

$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} dt + \int_1^\infty \frac{t^{n-1}}{(1+t)^{m+n}} dt$

$$I_1 = \int_1^{\infty} \frac{t^{n-1}}{(1+t)^{m+n}}dt$$

put $t = \dfrac{1}{z}$ $\quad\quad \because dt = \dfrac{-1}{z^2}dz$

$$= \int_1^{0} \frac{1}{z^{n-1}\left(1+\frac{1}{z}\right)^{m+n}} \cdot \left(\frac{-1}{z^2}\right)dz$$

$$= -\int_1^{0} \frac{1}{z^{n-1}\,\frac{(z+1)^{m+n}}{z^{m+n}}\,z^2}dz$$

$$I_1 = \int_0^{1} \frac{z^{m-1}}{(1+z)^{m+n}}dz$$

$$I_1 = \int_0^{1} \frac{t^{m-1}}{(1+t)^{m+n}}dt$$

$$I_{m,n} = \int_0^{1} \frac{t^{n-1}}{(1+t)^{m+n}}dt + \int_0^{1} \frac{t^{m-1}}{(1+t)^{m+n}}dt$$

$$I_{m,n} = \int_0^{1} \frac{t^{n-1}+t^{m-1}}{(1+t)^{m+n}}dt$$

$$I_{m,n} = \int_0^{1} \frac{x^{n-1}+x^{m-1}}{(1+x)^{m+n}}dx$$

$\therefore \ \alpha = 1$

10. If the arithmetic mean and geometric mean of the p^{th} and q^{th} terms of the sequence – 16, 8, – 4, 2, ... satisfy the equation $4x^2 - 9x + 5 = 0$, then p + q is equal to ________.

Ans. Official Answer NTA : (10)

Sol. $4x^2 - 8x + 5 = 0$

$x = 1$ or $x = \dfrac{5}{4}$

$\because$ A.M > G.M

$\therefore$ A.M $= \dfrac{5}{4}$ and G.M = 1

Let p^{th} term = a

q^{th} term = b

$\frac{a+b}{2} = \frac{5}{4}$ and $\sqrt{ab} = 1$

$\Rightarrow a + b = \frac{5}{2}$ and $ab = 1$

$\therefore\ a = (-16)\left(-\frac{1}{2}\right)^{p-1}$ and $b = (-16)\left(-\frac{1}{2}\right)^{q-1}$

$(-16)\left(-\frac{1}{2}\right)^{p-1} \times (-16)\left(-\frac{1}{2}\right)^{q-1} = 1$

$\left(-\frac{1}{2}\right)^{p+q-2} = \left(\frac{1}{2}\right)^{8}$

$p + q - 2 = 8$

$p + q = 10$