JEE Main July 2021

Question Paper With Text Solution

25 July. | Shift-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911
Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE MAIN JULY 2021 | 25TH JULY SHIFT-1

SECTION – A

1. Let $f(x) = 3\sin^4 x + 10\sin^3 x + 6\sin^2 x - 3$, $x \,\varepsilon \left[-\frac{\pi}{6}, \frac{\pi}{2}\right]$. Then, f is :

(1) decreasing in $\left(0, \frac{\pi}{2}\right)$

(2) decreasing in $\left(-\frac{\pi}{6}, 0\right)$

(3) increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(4) increasing in $\left(-\frac{\pi}{6}, \frac{\pi}{2}\right)$

Ans. Official Answer NTA (2)

Sol. $f'(x) = 12\sin^3 + 30\sin^2 x + 12\sin x$

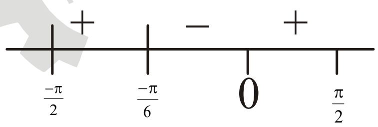
$= 6\sin x[2\sin^2 x + 5\sin x + 2\sin x]$

$= 6\sin x[(2\sin x + 1)(2\sin x + 2)]$

$= 6(\sin x + 2)(2\sin x + 1)(\sin x)$

$\sin x = 0, \quad \Rightarrow x = 0$

$\sin x = \frac{-1}{2}, \quad \Rightarrow x = \frac{-\pi}{6}$



increasing in $x \in \left(-\frac{\pi}{2}, -\frac{\pi}{6}\right) \cup \left(0, \frac{\pi}{2}\right)$

decreasing in $x \in \left(-\frac{\pi}{6}, 0\right)$

2. The value of the definite integral $\int_{\pi/24}^{5\pi/24} \frac{dx}{1+\sqrt[3]{\tan 2x}}$ is :

(1) $\frac{\pi}{6}$

(2) $\frac{\pi}{18}$

(3) $\frac{\pi}{12}$ (4) $\frac{\pi}{3}$

Ans. Official Answer NTA (3)

Sol. $I=\int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}\frac{dx}{1+(\tan 2x)^{1/3}}$ ....(1)

Apply king property $I=\int_a^b I(a+b-x)$

$I=\int_{\frac{\pi}{24}}^{\frac{5\pi}{24}}\frac{dx}{1+(\cot 2x)^{1/3}}$ ....(2)

equation (1) + (2)

$2I=\int_{\frac{\pi}{24}}^{\frac{5\pi}{24}} dx=\left(\frac{5\pi}{24}-\frac{\pi}{24}\right)=\frac{4\pi}{24}=\frac{\pi}{6}$

$I=\frac{\pi}{12}$

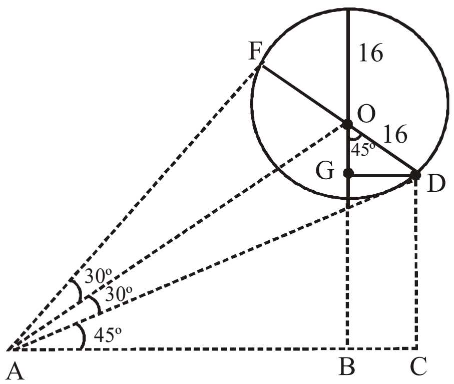
3. A spherical gas balloon of radius 16 meter subtends and angle 60° at the eye of the observer A while the angle of elevation of its center from the eye of A is 75°. Then the height (in meter) of the top most point of the balloon from the level of the observer's eye is :

(1) $8(2+2\sqrt{3}+\sqrt{2})$ (2) $8(\sqrt{6}-\sqrt{2}+2)$

(3) $8(\sqrt{6}+\sqrt{2}+2)$ (4) $8(\sqrt{2}+2+\sqrt{3})$

Ans. Official Answer NTA (3)

Sol.



$OG = 16\cos 45^\circ = 8\sqrt{2}$

$GD = BC = 16\cos 45^\circ = 8\sqrt{2}$

In ΔACD

$\sin 45^\circ = \frac{CD}{AD} = \frac{1}{\sqrt{2}}$

$AD = \sqrt{2}\,CD$

In ΔAOD

$\tan 30 = \frac{OD}{AD}$

$\frac{1}{\sqrt{3}} = \frac{16}{AD}$

$AD = 16\sqrt{3}$

$16\sqrt{3} = \sqrt{2}\ CD$

$CD = 8\sqrt{6} = BG$

$BE = BG + OG + OE$

$= 8\sqrt{6} + 8\sqrt{2} + 16$

$= 8[\sqrt{6} + \sqrt{2} + 2)$

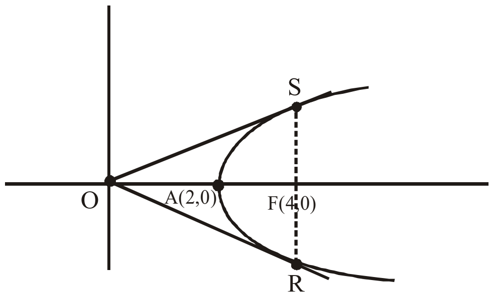
4. Let a parabola P be such that its vertex and focus lie on the positive x-axis at a distance 2 and 4 units from the origin, respectively. If tangents are drawn from O(0, 0) to the parabola P which meet P at S and R, then the area (in sq. units) of ΔSOR is equal to :

(1) 16 (2) 32

(3) $16\sqrt{2}$ (4) $8\sqrt{2}$

Ans. Official Answer NTA (1)

Sol.



Origin is foot of directrix, so tangents from 'O' meets at ends of 'LR'

Hence, points S and R are ends of LR

$S(4,4)$ $\qquad$ $R(4,-4)$

Area of $\Delta SOR = \frac{1}{2}(8)(4) = 16$

5. The number of real roots of the equation

$e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^{x} + 1 = 0$ is :

(1) 4 $\qquad$ (2) 6

(3) 1 $\qquad$ (4) 2

Ans. Official Answer NTA (4)

Sol. $e^{6x} - e^{4x} - 2e^{3x} - 12e^{2x} + e^{x} + 1 = 0$

$e^{6x} - 2e^{3x} + 1 - e^{4x} - 12e^{2x} + e^{x} = 0$

$(e^{3x}-1)^2 - e^{x}(e^{3x}-1) - 12e^{2x} = 0$

let $e^{3x} - 1 = p$ $\qquad$ $e^{x} = q$

$p^2 - pq - 12q^2 = 0$

$p^2(p-4q) + 39(p-42) = 0$

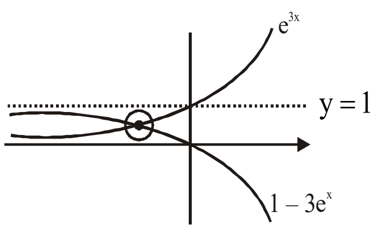
$p(9-4q) + 3q(9-4q) = 0$

$(p+3q)(p-4q) = 0$

case-1

$p + 3q = 0$

$e^{3x} - 1 + 3e^{x} = 0$



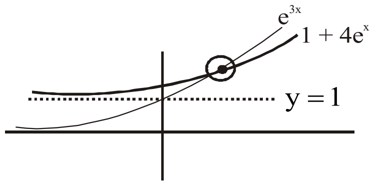
case-2

$p - 4q = 0$

$p = 4q$

$e^{3x} - 1 = 4e^{x}$

$e^{3x} = 1 + 4e^{x}$



6. Let y = y(x) be the solution of the differential equation

$\frac{dy}{dx} = 1 + x\,e^{y-x}$, $-\sqrt{2} < x < \sqrt{2}$, y(0) = 0

then, the minimum value of y(x), x ε $\left(-\sqrt{2}, \sqrt{2}\right)$ is equal to :

(1) $\left(2-\sqrt{3}\right) - \log_e 2$

(2) $\left(1-\sqrt{3}\right) - \log_e\left(\sqrt{3}-1\right)$

(3) $\left(2+\sqrt{3}\right) + \log_e 2$

(4) $\left(1+\sqrt{3}\right) - \log_e\left(\sqrt{3}-1\right)$

Ans. Official Answer NTA (2)

Sol. $e^{-y}.\frac{dy}{dx} - e^{-y} = xe^{-x}$

$-e^{-y} = t$

$e^{-y}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + t = xe^{-x}$

$IF = e^{\int dx} = e^x$

$t.e^x = \int x.e^{-x}.e^x dx$

$t.e^x = \frac{x^2}{2} + C$

$-e^{-y}e^x = \frac{x^2}{2} + C$

$\because y(0) = 0$

$-(1)(1) = 0 + c \Rightarrow c = -1$

$-e^{-y}e^x = \frac{x^2}{2} - 1$

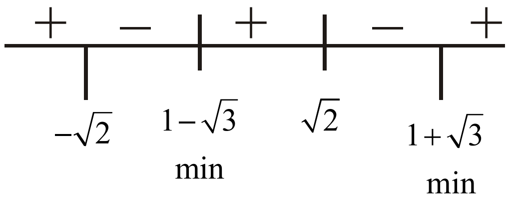
$e^{x-y} = \frac{2-x^2}{2}$

$$x - y = \ln\left(\frac{2-x^2}{2}\right)$$

$$y = x - \ln\left(\frac{2-x^2}{2}\right)$$

$$\frac{dy}{dx} = 1 - \frac{2}{2-x^2}\left(-\frac{2x}{2}\right) = 1 + \frac{2x}{2-x^2} = \frac{x^2-2-2x}{x^2-2}$$

$$\frac{dy}{dx} = \frac{(x-(1+\sqrt{3}))(x-(1-\sqrt{3}))}{(x-\sqrt{2})(x+\sqrt{2})}$$



$\because x \in (-\sqrt{2}, \sqrt{2})$

minima at $x = 1-\sqrt{3}$

$y = 1-\sqrt{3} - \ln(\sqrt{3}-1)$

7. Let the foot of perpendicular from a point P(1, 2, –1) to the straight line $L : \frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ be N. Let a line be drawn from P parallel to the plane x + y + 2z = 0 which meets L at point Q. If α is the acute angle between the lines PN and PQ, then cosα is equal to _____ .

(1) $\frac{1}{\sqrt{3}}$

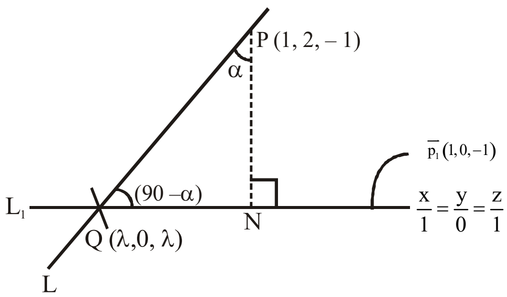
(2) $\frac{1}{\sqrt{5}}$

(3) $\frac{\sqrt{3}}{2}$

(4) $\frac{1}{2\sqrt{3}}$

Ans. Official Answer NTA (1)

Sol.



$\overrightarrow{PQ} \,||\, \text{Plane}$

$\overrightarrow{PQ}(\lambda - 1, -2, -\lambda + 1)$

$\overrightarrow{PQ}.\vec{n} = 0 \quad [\vec{n} = (1,1,2)]$

$\lambda - 1 - 2 - 2\lambda + 2 = 0$

$-\lambda - 1 = 0$

$Q(-1, 0, -1)$

$\overrightarrow{PQ} = (-2, -2, 2)$

$$\cos(90 - \alpha) = \frac{\overrightarrow{PQ}.\overrightarrow{P_1}}{|\overrightarrow{PQ}|.|\overrightarrow{P_1}|}$$

$$\sin\alpha \left|\frac{-2+0-2}{\sqrt{4+4+4}\sqrt{2}}\right| = \frac{4}{2\sqrt{3}\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\cos\alpha = \frac{1}{\sqrt{3}}$$

8. Let an ellipse E : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a^2 > b^2$, passes through $\left(\sqrt{\frac{3}{2}}, 1\right)$ and has eccentricity $\frac{1}{\sqrt{3}}$. If a circle, centered at focus $F(\alpha, 0)$, $\alpha > 0$, of E and radius $\frac{2}{\sqrt{3}}$, intersects E at two points P and Q, then PQ^2 is equal to :

(1) 3 (2) $\frac{8}{3}$

(3) $\frac{4}{3}$ (4) $\frac{16}{3}$

Ans. Official Answer NTA (4)

Sol. Ellipse passing through $\left(\frac{\sqrt{3}}{\sqrt{2}}, 1\right)$

$$\frac{3}{2a^2} + \frac{1}{b^2} = 1 \qquad ....(1)$$

$$e^2 = 1 - \frac{b^2}{a^2}$$

$$\frac{1}{3} = 1 - \frac{b^2}{a^2}$$

$\frac{b^2}{a^2}=\frac{2}{3}\Rightarrow\frac{3}{2a^2}=\frac{1}{b^2}$ ....(2)

from (1) & (2)

$b=\sqrt{2}$ $a=\sqrt{3}$

$F(ac,0)$ $=F(1,0)=$ centre (1, 0), Radius $=\frac{2}{\sqrt{3}}$

$E:\frac{x^2}{3}+\frac{y^2}{2}=1$

circle $(x-1)^2+y^2=\frac{4}{3}$

point of intersection

$P\left(1,\frac{2}{\sqrt{3}}\right)$ $Q\left(1,-\frac{2}{\sqrt{3}}\right)$

$PQ=\frac{16}{3}$

9. The locus of the centroid of the triangle formed by any point P on the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, and its foci is :

(1) $16x^2 - 9y^2 + 32x + 36y - 36 = 0$

(2) $9x^2 - 16y^2 + 36x + 32y - 36 = 0$

(3) $16x^2 - 9y^2 + 32x + 36y - 144 = 0$

(4) $9x^2 - 16y^2 + 36x + 36y - 144 = 0$

Ans. Official Answer NTA (1)

Sol. $16(x+1)^2-a(y-2)^2=144$

$\frac{(x+1)^2}{9}-\frac{(y-2)^2}{16}=1$

Let $P(-1+3\sec\theta,\ \ 2+4\tan\theta)$

$e=\sqrt{1+\frac{16}{9}}=\frac{5}{3}$

$ae=5$

$F_1(4,2),\ \ F_2(-6,2)$

Let centroid of ΔF_1F_2P is (h,k)

$h = \frac{4-6-1+3\sec\theta}{3}$ $\quad$ $k = \frac{2+2+2+4\tan\theta}{3}$

$h = -1+\sec\theta$ $\quad$ $\left(\frac{3k-6}{4}\right) = \tan\theta$

$\sec\theta = h+1$

$\sec^2\theta - \tan^2\theta = 1$

$(h+1)^2 - \left(\frac{3k-6}{4}\right)^2 = 1$

$16(x+1)^2 - (3y-6)^2 = 16$

10. Let S_n be the sum of the first n terms of an arithmetic progression. If $S_{3n} = 3S_{2n}$, then the value of $\frac{S_{4n}}{S_{2n}}$ is :

(1) 8 $\quad$ (2) 4

(3) 2 $\quad$ (4) 6

Ans. Official Answer NTA (4)

Sol. $S_{3n} = 3S_{2n}$

$\frac{3n}{2}[2a+(3n-1)d] = \frac{3.(2n)}{2}[2a+(2n-1)d]$

$2a+(3n-1)d = 4a+(4n-2)d$

$(3n-1-4n+2)d = 2a$

$(1-n)d = 2a$

$$\frac{S_{4n}}{S_{2n}} = \frac{\frac{4n}{2}[2a+(4n-1)d]}{\frac{2n}{2}[2a+(2n-1)d]}$$

$= 2\frac{[(1-n)d+(4n-1)d]}{[(1-n)d+(2n-1)d]}$

$= 2(3) = 6$

11. Let 9 distinct balls be distributed among 4 boxes, B_1, B_2, B_3 and B_4. If the probability that B_3 contains exactly 3 balls is $k\left(\frac{3}{4}\right)^9$ then k lies in the set :

(1) $\{x \in R : |x - 3| < 1\}$

(2) $\{x \in R : |x - 2| \le 1\}$

(3) $\{x \in R : |x - 1| < 1\}$

(4) $\{x \in R : |x - 5| \le 1\}$

Ans. Official Answer NTA (1)

Sol. $p(\in) = \frac{{}^9C_3(3)^6}{(4)^9} = \left(\frac{28}{9}\right)\left(\frac{3}{4}\right)^9$

$k = \frac{28}{9}$

12. Let $g : N \to N$ be defined as

$g(3n + 1) = 3n + 2,$

$g(3n + 2) = 3n + 3,$

$g(3n + 3) = 3n + 1$, for all $n \ge 0$.

Then which of the following statements is true?

(1) There exists an onto function $f : N \to N$ such that $fog = f$

(2) $gogog = g$

(3) There exists a function $f : N \to N$ such that $gof = f$

(4) There exists a one-one function $f : N \to N$ such that $fog = f$

Ans. Official Answer NTA (1)

Sol. $g(3n + 1) = 3n + 2$

$g(3n + 2) = 3n + 3$

$g(3n + 3) = 3n + 1$, n ? 0

For $x = 3n + 1$

(1) $gogog(3n + 1) = gog(3n + 2) = g(3n + 3) = 3n + 1$

Similarly

$gogog(3n + 2) = 3n + 2$

gogog (3n + 3) = 3n + 3

So gogog (x) = x $\forall x \in N$

(2) As f : N $\to$ N, f = 3n + 1

= 3n + 2

= 3n + 3

So, g(3n + 1) = 3n + 2, g(3n + 2) = 3n + 3, g(3n + 3) = 3n + 1

So g(f(x)) $\neq$ f(x)

(3) If f : N $\to$ N and f is a one-one function such that f(g(x)) = f(x) then

g(x) = x

but g(x) $\neq$ x

(4) If f : N $\to$ N and f is an onto function such that f(g(x)) = f(x) then

One of its possibilities is by taking f(x) as onto function

$$f(x) = \begin{cases} a & x = 3n+1 \\ a & x = 3n+2, \quad a \in N \\ a & x = 3n+3 \end{cases}$$

$\Rightarrow f(g(x)) = f(x) \quad \forall x \in N$

13. Let the vectors

$(2 + a + b)\hat{i} + (a + 2b + c)\hat{j} - (b + c)\hat{k}$,

$(1 + b)\hat{i} + 2b\hat{j} - b\hat{k}$ and $(2 + b)\hat{i} + 2b\hat{j} + (1 - b)\hat{k}$, a, b, c $\in$ R

be co-planar. Then which of the following is true?

(1) 2b = a + c (2) a = b + 2c

(3) 3c = a + b (4) 2a = b + c

Ans. Official Answer NTA (1)

Sol. vectors are co-planer then S.T.P = 0

$$\begin{vmatrix} 2+a+b & a+2b+c & -b-c \\ 1+b & 2b & -b \\ 2+b & 2b & 1-b \end{vmatrix} = 0$$

$R_3 \to R_3 - R_2$

$$\begin{vmatrix} 2+a+b & a+2b+c & -b-c \\ 1+b & 2b & -b \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_3$

$$\begin{vmatrix} 2+a+2b+c & a+2b+c & -b-c \\ 1+2b & 2b & -b \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$

$$\begin{vmatrix} 2 & a+2b+c & -b-c \\ 1 & 2b & -b \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$4b - a - 2b - c = 0$

$2b = a + c$

14. The values of a and b, for which the system of equations

$2x + 3y + 6z = 8$

$x + 2y + az = 5$

$3x + 5y + 9z = b$

has no solution, are :

(1) $a = 3, b = 13$ (2) $a \neq 3, b \neq 13$

(3) $a \neq 3, b = 3$ (4) $a = 3, b \neq 13$

Ans. Official Answer NTA (4)

Sol. $$D = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 2 & a \\ 3 & 5 & 9 \end{vmatrix} = 0$$

$\Rightarrow 2(18 - 5a) - 3(9 - 3a) + 6(5 - 6) = 0$

$36 - 10a - 27 + 9a - 6 = 0$

$-a + 3 = 0$

$$D_x = \begin{vmatrix} 3 & 3 & 6 \\ 5 & 2 & 3 \\ b & 5 & a \end{vmatrix}$$

$= 8(3) - 5(-3) + b(9 - 12)$

$= 24 + 15 - 3b$

$= 3(13 - b)$

$$D_y = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & 3 \\ 3 & b & 9 \end{vmatrix} = 0$$

$$D_z = \begin{vmatrix} 2 & 3 & 8 \\ 1 & 2 & 5 \\ 3 & 5 & b \end{vmatrix} = (b - 13)$$

$b \neq 13$

15. The Boolean expression

$(p \Rightarrow q) \wedge (q \Rightarrow \sim P)$ is equivalent to :

(1) P (2) ~ q

(3) ~ p (4) q

Ans. Official Answer NTA (3)

Sol. $(p \to q) \wedge (q \to \sim p)$

p	q	~ p	$p \to q$	$q \to \sim p$	$(p \to q) \wedge (q \to \sim p)$
T	T	F	T	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T

16. The sum of all values of x in [0, 2π], for which

sinx + sin2x + sin3x + sin4x = 0, is equal to :

(1) 9 π (2) 11 π

(3) 12 π (4) 8 π

Ans. Official Answer NTA (1)

Sol. $\sin x + \sin 2x + \sin 3x + \sin 4x = 0$

$$\frac{\sin 2x.\sin\left(\frac{5x}{2}\right)}{\sin\left(\frac{x}{2}\right)}=0$$

Case-I

$\sin 2x = 0$

$x = 0, \frac{\pi}{2}, \pi, 3\frac{\pi}{2}, 2\pi$

Case-II

$\sin\frac{5x}{2} = 0$

$x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{2\pi}{5}, 2\pi$

solution of $\sin\frac{x}{2} = 0$ are valid, because $x = 0, \& \ 2\pi$ satisfy the given equation.

So solutions are $x = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{2\pi}{5}, 2\pi, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

17. Let f : $[0, \infty) \to [0, \infty)$ be defined as

$f(x) = \int_0^x [y]\,dy$

where [x] is the greatest integer less than or equal to x. Which of the following is true?

(1) f is differentiable at every point in $[0, \infty)$.

(2) f is continuous everywhere except at the integer points in $[0, \infty)$.

(3) f is continuous at every point in $[0, \infty)$ and differentiable except at the integer points.

(4) f is both continuous and differentiable except at the integer points in $[0, \infty)$.

Ans. Official Answer NTA (3)

Sol. let $n \le x < n+1$

$f(x) = \int_0^1 (0)\,dy + \int_1^1 (1)\,dy + \int_2^3 2dy ...... + \int_n^x (n)\,dy$

$f(x) = 0 + 1 + 2(1) + 3(1).....(n-1)(1) + n(x-n)$

$= 1 + 2 + 3 .... + n - 1 + nx - n^2$

$= \frac{(n-1)(n)}{2} + nx - n^2$

$= n\left(x - \frac{n+1}{2}\right)$

$$f(x) = \begin{cases} x\left(\frac{x-1}{2}\right) & x \in n\,(\text{integer}) \\ [x]\left(\frac{2x-[x]-1}{2}\right) & x\,(\text{integer}) \end{cases}$$

$\lim_{x \to n} f(x) = \frac{n(n-1)}{2}$ continuous

$f(x) = \int_0^x [y]\,dy$

$f'(x) = [x]$

ND at all integers

18. If b is very small as compared to the value of a, so that the cube and other higher powers of $\frac{b}{a}$ can be neglected in the identity

$\frac{1}{a-b} + \frac{1}{a-2b} + \frac{1}{a-3b} + ... + \frac{1}{a-nb} = \alpha n + \beta n^2 + \gamma n^3,$

then the value of γ is :

(1) $\frac{a^2+b}{3a^3}$ (2) $\frac{a+b^2}{3a^3}$

(3) $\frac{a+b}{3a^3}$ (4) $\frac{b^2}{3a^3}$

Ans. Official Answer NTA (4)

Sol. $\frac{1}{a}\left(\frac{1}{1-\frac{b}{a}} + \frac{1}{1-\frac{2b}{a}} + \frac{1}{1-\frac{3b}{a}} .... + \frac{1}{1-\frac{nb}{a}}\right)$

$= \frac{1}{a}\left[\left(1-\frac{b}{a}\right)^{-1} + \left(1-\frac{2b}{a}\right)^{-1} + \left(1-\frac{3b}{a}\right)^{-1} .... + \left(1-\frac{nb}{a}\right)\right]$

$= \frac{1}{a}\left[\left(1-\frac{b}{a}+\left(\frac{b}{a}\right)^2\right) + \left(1+\left(\frac{2b}{a}\right)+\left(\frac{2b}{a}\right)^2 .... \left(1+\left(\frac{nb}{a}\right)+\left(\frac{nb}{a}\right)^2\right)\right)\right]$

$\because \left(\frac{b}{a}\right)^n = 0 \qquad n \geq 3$

$= \frac{1}{a}\left[n + \frac{b}{a}(1+2+3.....n) + \left(\frac{b}{a}\right)^2 [1^2 + 2^2......n^2]\right]$

$= \frac{1}{a}\left[n + \frac{b}{a}\frac{n(n+1)}{2} + \frac{b^2}{a^2}\frac{n(n+1)(2n+1)}{6}\right]$

$= \frac{n}{a} + \frac{b}{2a^2}[n^2+n] + \frac{b^2}{6a^3}(2n^3+3n^2+n)$

$= \left(\frac{1}{a} + \frac{b}{2a^2} + \frac{b^2}{6a^3}\right)n + \left(\frac{b}{2a^2} + \frac{b^2}{2a^3}\right)n^2 + \left(\frac{b^2}{3a^3}\right)n^3$

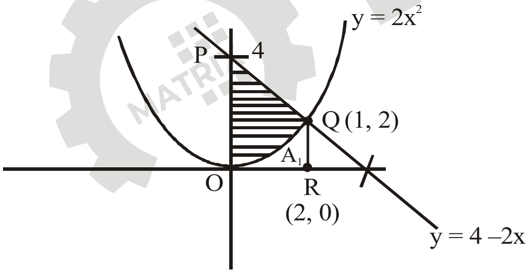
19. The area (in sq. units) of the region, given by the set $\{(x, y) \in R \times R \mid x \geq 0, 2x^2 \leq y \leq 4 - 2x\}$ is :

(1) $\frac{13}{3}$ (2) $\frac{17}{3}$

(3) $\frac{7}{3}$ (4) $\frac{8}{3}$

Ans. Official Answer NTA (3)

Sol.



$A_1 = \int_0^1 2x^2 dx = \left(\frac{2}{3}x^3\right)_0^1 = \frac{2}{3}$

Req Area = Area of trapezium ORQP – A_1

$= \frac{1}{2}(4+2)(1) - \frac{2}{3}$

$= 3 - \frac{2}{3} = \frac{7}{3}$

20. Let $f : R \to R$ be defined as

$$f(x)=\begin{cases}\dfrac{\lambda|x^2-5x+6|}{\mu(5x-x^2-6)}, & x<2\\ e^{\frac{\tan(x-2)}{x-[x]}} & ,x>2\\ \mu & ,x=2\end{cases}$$

where [x] is the greatest integer less than or equal to x. If f is continuous at x = 2, then $\lambda + \mu$

(1) 1 (2) 2e – 1

(3) e(e – 1) (4) e(–e + 1)

Ans. Official Answer NTA (4)

Sol. $f(x)$ is continuous at $x = 2$

$f(2^-) = f(2) = f(2^+)$

$$\lim_{x\to 2^-}\frac{\lambda|x^2-5x+6|}{-(x^2-5x+6)\mu}=\mu=\lim_{x\to 2^+}e^{\frac{\tan(x-2)}{x-[x]}}$$

$$-\frac{\lambda}{\mu}=\mu=e$$

$\mu = e,\quad \lambda = -e^2$

$\lambda + \mu = e(-e+1)$

SECTION - B

1. If α, β are roots of the equation $x^2 + 5(\sqrt{2})x + 10 = 0$, $\alpha > \beta$ and $P_n = \alpha^n - \beta^n$ for each positive integer n, then the value of $\left(\dfrac{P_{17}P_{20}+5\sqrt{2}P_{17}P_{19}}{P_{18}P_{19}+5\sqrt{2}P_{18}^2}\right)$ is equal to _______ .

Ans. Official Answer NTA (1)

Sol. $x^2+5\sqrt{2}x+10=0 <^{\alpha}_{\beta}$

$\alpha^2+5\sqrt{2}\alpha=-10 \qquad \& \ P_n=\alpha^n-\beta^n$

$\beta^2+5\sqrt{2}\beta=-10$

$$\Rightarrow \frac{P_{17}}{P_{18}}\left[\frac{P_{20}+5\sqrt{2}P_{19}}{P_{19}+5\sqrt{2}P_{18}}\right]$$

$$\Rightarrow \frac{P_{17}}{P_{18}}\left[\frac{\alpha^{20}-\beta^{20}+5\sqrt{2}(\alpha^{19}-\beta^{19})}{\alpha^{19}-\beta^{19}+5\sqrt{2}(\alpha^{18}-\beta^{18})}\right]$$

$$\Rightarrow \frac{P_{17}}{P_{18}}\left[\frac{\alpha^{18}(\alpha^2+5\sqrt{2}\alpha)-\beta^{18}(\beta^2+5\sqrt{2}\beta)}{\alpha^{17}(\alpha^2+5\sqrt{2}\alpha)-\beta^{17}(\beta^2+5\sqrt{2}\beta)}\right]$$

$$\Rightarrow \frac{P_{17}}{P_{18}}\left[\frac{(-10)(\alpha^{18}-\beta^{18})}{(-10)(\alpha^{17}-\beta^{17})}\right]$$

$$\Rightarrow \frac{P_{17}}{P_{18}}\times\frac{P_{18}}{P_{17}}=1$$

2. Let $S=\left\{n\in N\left|\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\forall a,b,c,d\in R\right.\right\}$, where $i=\sqrt{-1}$. Then the number of 2-digit numbers in the set S is _____ .

Ans. Official Answer NTA (11)

Sol. $S=\left\{n\in N\left|\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}^n\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\forall a,b,c,d\in R\right.\right.$

let $A=\begin{bmatrix}0 & i\\ 1 & 0\end{bmatrix}$

$$A^2=\begin{bmatrix}0 & i\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & i\\ 1 & 0\end{bmatrix}=\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}$$

$$A^4=\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}\begin{bmatrix}i & 0\\ 0 & i\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

$$A^8=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

$A^{16}=A^{24}=A^{32}.............=A^{96}=I$

No. of two digits numbers = 11

3. Let $M = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \{\pm3, \pm2, \pm1, 0\} \right\}$. Define f : M → Z, as f(A) = det(A), for all A ∈ M, where Z is set of all integers. Then the number of A ∈ M such that f(A) = 15 is equal to _____ .

Ans. Official Answer NTA (16)

Sol. $f(A) = \det(A) = 15$

$ad - bc = 15$

case -1

$ad = 9 \quad \& \quad bc = -6$

$ad = (3,3)$ or $(-3,-3)$ $bc = (2,-3), (-2,3), (-3,2), (3,-2)$

No. of ways = $(2)\times(4) = 8$

case -2

$ad = 6 \quad \& \quad bc = -9$

ad = (2, 3) (3, 2) (–2, –3) (–3, –2) and bc = (3, –3) (–3, 3)

No. of ways = $4 \times 2 = 8$

Total no. of ways = 8 + 8 = 16

4. The term independent of 'x' in the expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right)^{10}$, where x ≠ 0, 1 is equal to ______ .

Ans. Official Answer NTA (210)

Sol. $\left(\left(x^{1/3} + 1\right) - \frac{(\sqrt{x} + 1)}{\sqrt{x}} \right)^{10}$

$\left(x^{\frac{1}{3}} - \frac{1}{x^{\frac{1}{2}}} \right)^{10}$

$T_{r+1} = {}^{10}C_r (x)^{\frac{10-r}{3}} (x)^{-\frac{r}{2}}$

$= {}^{10}C_r (x)^{\frac{10-r}{3} - \frac{r}{2}}$

$\frac{10-r}{3} - \frac{r}{2} = 0$

$20 - 2r - 3r = 0$

$r = 4$

T_5 is independent of 'x' $= {}^{10}C_4 = 210$

5. The ratio of the coefficient of the middle term in the expansion of $(1+x)^{20}$ and the sum of the coefficients of two middle terms in expansion of $(1+x)^{19}$ is _________.

Ans. Official Answer NTA (1)

Sol. $$\frac{{}^{20}C_{10}}{{}^{19}C_9 + {}^{19}C_{10}} = \frac{{}^{20}C_{10}}{{}^{20}C_{10}} = 1$$

6. If the value of $\left(1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\text{......... upto}\infty\right)^{\log_{(0.25)}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\text{........upto}\infty\right)}$ is l, then l^2 is equal to ________.

Ans. Official Answer NTA (3)

Sol. Let

$$S_1 = 1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}..........$$

$$\frac{S_1}{3} = \frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}..........$$

$$\frac{2S_1}{3} = 1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}..........$$

$$\frac{2S_1}{3} = \frac{4}{3}+\frac{4}{3^2}+\frac{4}{3^3}..........$$

$$\frac{2S_1}{3} = \frac{4}{3}\left[\frac{1}{1-\frac{1}{3}}\right]$$

$$\frac{2S_1}{3} = \frac{4}{3}\times\frac{3}{2}$$

$$S_1 = 3$$

$$S_2 = \frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}...........$$

$S_2 = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$

$\ell = (S_1)^{\log_{0.25}(S_2)} = (3)^{\log_{0.25}(0.5)} = \sqrt{3}$

$\ell^2 = 3$

7. Let $\vec{p} = 2\hat{i}+3\hat{j}+\hat{k}$ and $\vec{q} = \hat{i}+2\hat{j}+\hat{k}$ be two vectors. If a vector $\vec{r} = (\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k})$ is perpendicular to each of the vectors $(\vec{p}+\vec{q})$ and $(\vec{p}-\vec{q})$, and $|\vec{r}| = \sqrt{3}$, then $|\alpha|+|\beta|+|\gamma|$ is equal to _________.

Ans. Official Answer NTA (3)

Sol. $\vec{p}+\vec{q} = (3,5,2)$

$\vec{p}-\vec{q} = (1,1,0)$

$\vec{r} = (\alpha,\beta,\gamma)$

$\vec{r}.(\vec{p}+\vec{q}) = 0$

$3\alpha+5\beta+2\gamma = 0$

$\vec{r}.(\vec{p}-\vec{q}) = 0$

$\alpha+\beta = 0$

$\beta = -\alpha$

$\gamma = \alpha$

$\vec{r} = \alpha\hat{i}-\alpha\hat{j}+\alpha\hat{k}$

$|\vec{r}| = \sqrt{3\alpha^2} = \sqrt{3}$

$\alpha^2 = 1 \Rightarrow \alpha = \pm 1$

$|\alpha|+|\beta|+|\gamma| = 3$

8. Consider the following frequency distribution :

Class:	$10-20$	$20-30$	$30-40$	$40-50$	$50-60$
Frequency:	α	110	54	30	β

If the sum of all frequencies is 584 and median is 45, then $|\alpha-\beta|$ is equal to __________.

Ans. Official Answer NTA (164)

Sol.

class	frequency	cumulative freq.
$10-20$	α	α
$20-30$	110	$\alpha+110$
$30-40$	54	$\alpha+164$
$40-50$	30	$\alpha+194$
$50-60$	β	$\alpha+\beta+194=584$

$N = 584$

$\alpha+\beta = 390$

$\frac{N}{2} = 292$

Median $= 45 = \ell + \left[\frac{\frac{N}{2}-F}{f}\right) \times 4$

$45 = 40 + \left[\frac{292-(\alpha+164)}{30}\right]$

$\alpha = 113$

$\beta = 277$

9. There are 5 students in class 10, 6 students in class 11 and 8 students in class 12. If the number of ways, in which 10 students can be selected from them so as to include at least 2 students from each class and at most 5 students from the total 11 students of class 10 and 11 is 100 k, then k is equal to ________.

Ans. Official Answer NTA (238)

Sol.

$10^{th}(5)$	$11^{th}(6)$	$12^{th}(8)$	No.of ways
2	2	6	${}^5C_2.{}^6C_2.{}^8C_6$
2	3	5	${}^5C_2{}^6C_3.{}^8C_5$
3	2	5	${}^5C_3.{}^6C_2.{}^8C_5$

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10. Let $y = y(x)$ be solution of the follwoing differential equation

$$e^y \frac{dy}{dx} - 2e^y \sin x + \sin x \cos^2 x = 0, y\left(\frac{\pi}{2}\right) = 0.$$

If $y(0) = \log_e(\alpha + \beta e^{-2})$, then $4(\alpha + \beta)$ is equal to _________.

Ans. Official Answer NTA (4)

Sol. $e^y \frac{dy}{dx} - 2e^y \sin x = -\sin x \cos^2 x$

Put $e^y = t$

$e^y \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - 2t \sin x = -\sin x \cos^2 x$

$\text{I.F.} = e^{-\int 2\sin x dx} = e^{2\cos x}$

$e^y . e^{2\cos x} = \int e^{2\cos x}(-\sin x\_\cos^2 x dx$

$\cos x = z$

$= \int p^{2z} z^2 dz$

$\frac{e^{2z}}{2} z^2 - \int e^{2z}.zdz$

$= \frac{e^{2z}}{2} z^2 - \int e^{2z}.zdz$

$= \frac{e^{2z}}{2} z^2 - \left[\frac{e^{2z}}{2}.z - \frac{e^{2z}}{4}\right] + c$

$\Rightarrow e^y e^{2\cos x} = \frac{e^{2z}}{4}(2z^2 - 2z + 1) + c$

$\Rightarrow e^y e^{2\cos x} = \frac{e^{2\cos x}}{4}(2\cos^2 x - 2\cos x + 1) + c$

$\Rightarrow e^y = \frac{1}{4}(2\cos^2 x - 2\cos x + 1) + c$

At $x = \frac{\pi}{2} y = 0,$

$\Rightarrow 1 = \frac{1}{4} + c \Rightarrow c = \frac{3}{4}$

$\Rightarrow e^{y}=\frac{1}{4}(2\cos^{2}x-2\cos x+1)+\frac{3}{4}e^{-2\cos x}$

$\Rightarrow y(0)=\ell n\left(\frac{1}{4}+\frac{3}{4}e^{-2}\right)$

$\Rightarrow y(0)=\ell n(\alpha+\beta e^{-2})$

$\Rightarrow \alpha=\frac{1}{4},\beta=\frac{3}{4}$

$\Rightarrow 4(\alpha+\beta)=4$