JEE Main July 2021

Question Paper With Text Solution

25 July. | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JULY 2021 | 25TH JULY SHIFT-2

SECTION – A

1. The first of the two samples in a group has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and standard deviaton $\sqrt{13.44}$, then the standard deviation of the second sample is :

(1) 6

(2) 4

(3) 5

(4) 8

Ans. Official Answer NTA (2)

Sol. $15.6 = \frac{100 \times 15 + 150 \times \bar{x}}{250}$

$\bar{x} = 16$

Combined satndard deviation $= \sqrt{13.44}$

$\Rightarrow$ Combined variance $(\sigma^2) = 13.44$

$\sigma^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2$

$13.44 = \frac{\Sigma x_i^2}{250} - 243.36$

$\Rightarrow \Sigma x_i^2 = 64200$

For 1st sample

$9 = \frac{\Sigma x_I^2}{100} - 225$

$\Rightarrow \Sigma x_I^2 = 23400$

For 2nd sample

$\Sigma x_{II}^2 = 64200 - 23400 = 40800$

standard deviation of IInd sample will be

$\sqrt{\frac{\Sigma x_{II}^2}{n} - (\bar{x}_{II})^2} = \sqrt{\frac{40800}{150} - 256} = 4$

2. If $f(x)=\begin{cases}\int_0^x (5+|1-t|)\,dt, & x>2\\ 5x+1 & x\le 2\end{cases}$, then

(1) f(x) is not differentiable at x = 1

(2) f(x) is not continuous at x = 2

(3) f(x) is everywhere differentiable

(4) f(x) is continuous but not differentiable at x = 2

Ans. Official Answer NTA (4)

Sol. $f(x)=\begin{cases}\int_0^x \left(5+|1-t|\right)dt & x>2\\ 5x+1 & x\le 2\end{cases}$

$f(x)=\int_0^1 (5+1-t)\,dt+\int_1^x (5+t-1)\,dt$

$=\int_0^1 (6-t)\,dt+\int_1^x (t+4)\,dt$

$=6-\frac{1}{2}+\frac{x^2}{2}+4x-\frac{1}{2}-4$

$f(x)=\frac{11+x^2+8x-1-8}{2}=\frac{x^2+8x+2}{2}$

$f(x)=\begin{cases}\frac{x^2+8x+2}{2} & x>2\\ 5x+1 & x\le 2\end{cases}$

$f(2^+)=\frac{4+16+2}{2}=11$

$f(2^-)=11$

$f(2)=11$

Continous

$f'(x)=\begin{cases}\frac{2x+8}{2} & x>2\\ 5 & x<2\end{cases}$

$$f'(x)=\begin{cases}x+4 & x>2\\ 5 & x<2\end{cases}$$

$f'(2^+)=6 \qquad f'(2^-)=2$

Not differentiable

3. Let a, b and c be distinct positive numbers. If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$ and $c\hat{i}+c\hat{j}+b\hat{k}$ are co-planar, then c is equal to :

(1) $\dfrac{2}{\frac{1}{a}+\frac{1}{b}}$

(2) $\sqrt{ab}$

(3) $\dfrac{a+b}{2}$

(4) $\dfrac{1}{a}+\dfrac{1}{b}$

Ans. Official Answer NTA (2)

Sol. $$\begin{vmatrix}a & a & c\\ 1 & 0 & 1\\ c & c & b\end{vmatrix}=0$$

$a(-c)-a(b-c)+c(c)=0$

$c^2=ab$

$c=\sqrt{ab}$

4. Let X be a random variable such that the probability function of a distribution is given by $P(X=0)=\frac{1}{2}, P(X=j)=\frac{1}{3^j}(J=1,2,3,.......,\infty)$ Then the mean of the distribution and P(X is positive and even) respectively are :

(1) $\frac{3}{8}$ and $\frac{1}{8}$

(2) $\frac{3}{4}$ and $\frac{1}{9}$

(3) $\frac{3}{4}$ and $\frac{1}{16}$

(4) $\frac{3}{4}$ and $\frac{1}{8}$

Ans. Official Answer NTA (4)

Sol. Mean = $\Sigma x_i p_i = \sum_{r=0}^{\infty} r \times \frac{1}{3r} = \frac{3}{4}$

P (x is even) = $\frac{1}{3^2} + \frac{1}{3^4} + \dots\dots + \infty$

$= \frac{\frac{1}{9}}{1-\frac{1}{9}} = \frac{\frac{1}{9}}{8/9} = \frac{1}{8}$

5. If $|\vec{a}| = 2, |\vec{b}| = 5$ and $|\vec{a} \times \vec{b}| = 8$, then $|\vec{a}.\vec{b}|$ is equal to :

(1) 6

(2) 4

(3) 3

(4) 5

Ans. Official Answer NTA (1)

Sol. $\left|\overline{a} \times \overline{b}\right| = \left|\overline{a}\right|\left|\overline{b}\right| \sin\theta$

$8 = 2 \times 5 \times \sin\theta$

$\sin\theta = 4/5 = \frac{P}{H} \quad B = 3$

$\cos\theta = 3/5$

$\left|\vec{a} \cdot \vec{b}\right| = \left|\vec{a}\right|\left|\vec{b}\right| \cos\theta$

$= 2 \times 5 \times \frac{3}{5} = 6$

6. If ${}^nP_r = {}^nP_{r+1}$ and ${}^nC_r = {}^nC_{r-1}$, then the value of r is equal to :

(1) 4

(2) 3

(3) 1

(4) 2

Ans. Official Answer NTA (4)

Sol. $^{n}P_{r} = {}^{n}P_{r+1}$

$$\frac{\lfloor n}{\lfloor n-r} = \frac{\lfloor n}{\lfloor n-r-1}$$

$$\frac{1}{(n-r)\lfloor n-r-1} = \frac{1}{\lfloor n-r-1}$$

$n - r = 1 \Rightarrow n = 1 + r$

$^{n}C_{r} = {}^{n}C_{r-1}$

$$\frac{\lfloor n}{\lfloor r \lfloor n-r} = \frac{\lfloor n}{\lfloor r-1 \lfloor n-r+1}$$

$$\frac{1}{r\lfloor r-1 \lfloor n-r} = \frac{1}{\lfloor r-1\,(n-r+1)\lfloor n-r}$$

$$\frac{1}{r} = \frac{1}{n-r+1}$$

$n - r + 1 = r$

$n + 1 = 2r$

$1 + r + 1 = 2r$

$r = 2$

7. The value of $\cot\frac{\pi}{24}$ is :

(1) $\sqrt{2}+\sqrt{3}+2+\sqrt{6}$

(2) $\sqrt{2}+\sqrt{3}+2-\sqrt{6}$

(3) $\sqrt{2}-\sqrt{3}-2+\sqrt{6}$

(4) $3\sqrt{2}-\sqrt{3}-\sqrt{6}$

Ans. Official Answer NTA (1)

Sol. $\cot\left(\frac{\pi}{29}\right) = \cot(7.5^o)$

$$\cot(7\cdot5^o) = \frac{1+\cos15^o}{\sin15^o}$$

$\cos15^o = \cos(45^o - 30^o) = \cos45^o\cos30^o + \sin45^o\sin30^o$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\times\frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cot(7.5^\circ) = \frac{2\sqrt{2}+\sqrt{3}+1}{\sqrt{3}-1}$

$\cot(7\cdot 5^\circ) = \frac{\left(2\sqrt{2}+\sqrt{3}+1\right)\left(\sqrt{3}+1\right)}{2}$

$= 2\sqrt{6}+2\sqrt{2}+3+\sqrt{3}+\sqrt{3}+1$

$\frac{2\sqrt{6}+2\sqrt{3}+2\sqrt{2}+4}{2}$

$= \sqrt{6}+\sqrt{3}+\sqrt{2}+2$

8. The lowest integer which is greater than $\left(1+\frac{1}{10^{100}}\right)^{10^{100}}$ is __________.

(1) 2

(2) 1

(3) 3

(4) 4

Ans. Official Answer NTA (3)

Sol. $\left(1+\frac{1}{10^{100}}\right)^{10^{100}}$

$x = 10^{100}$

$P = \left(1+\frac{1}{x}\right)^{x}$

$\left(1+\frac{1}{x}\right)^{x} = 1+1+\frac{x(x-1)}{\lfloor 2 x^2} + \frac{x(x-1)(x-2)}{x^3 \lfloor 3} + \ldots\ldots$

$\left(1+\frac{1}{x}\right)^{x} = 2 + \left(\frac{1}{\lfloor 2} - \frac{1}{\lfloor 2 x^2}\right) + \left(\frac{1}{\lfloor 3} - \right) + \ldots..$

$P = 2 + \left(\text{Positive values less than } \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \infty\right)$

$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} +$

$e - 2 = \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} +$

$P = 2 + e - 2$

$P = e \left(e \in (2,3)\right)$

Lowest integer greater than P = 3

9. The value of the integral $\int_{-1}^{1} \log(x + \sqrt{x^2+1})\,dx$ is :

(1) 0

(2) 2

(3) -1

(4) 1

Ans. Official Answer NTA (1)

Sol. $I = \int_{-1}^{1} \log\left(x + \sqrt{x^2+1}\right)dx$

$I = \int_{0}^{1} \left(\log(x + \sqrt{x^2+1}) + \log(-x + \sqrt{x^2+1})\right)dx$

$= \int_0^1 \log\left(x^2 + 1 - x^2\right)dx = 0$

10. If the greatest value of the term independent of 'x' in the expansion of $\left(x\sin\alpha + a\frac{\cos\alpha}{x}\right)^{10}$ is $\frac{10!}{(5!)^2}$,

then the value of 'a' is equal to :

(1) 1

(2) -2

(3) 2

(4) -1

Ans. Official Answer NTA (3)

Sol. $T_{r+1} = {}^{10}C_r \left(x\sin\alpha\right)^{10-r}\left(\frac{a\cos\alpha}{x}\right)^r$

$T_{r+1} = {}^{10}C_r (\sin\alpha)^{10-r}\left(a\cos\alpha\right)^r x^{10-r-r}$

$10-2r=0=r=5$

$T_{5+1} = {}^{10}C_5\left(\sin\alpha\right)^5 \quad \left(a\cos\alpha\right)^5$

$= {}^{10}C_5(\sin\alpha)^5\ a^5\ (\cos\alpha)^5$

$= {}^{10}C_5 \frac{1}{2^5}\times\left(\sin 2\alpha\right)^5 \times a^5$

$\left(\frac{a}{2}\right)^5 \times {}^{10}C_5 = \frac{\lfloor 10}{(\lfloor 5)^2}$

$\Rightarrow a = 2$

11. If a tangent to the ellipse $x^2+4y^2=4$ meets the tangents at the extremities of its major axis at B and C, then the circle with BC as diameter passes through the point :

(1) $(\sqrt{3},0)$

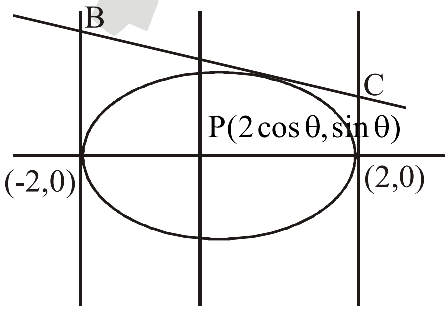
(2) (1, 1)

(3) (−1, 1)

(4) $(\sqrt{2}, 0)$

Ans. Official Answer NTA (1)

Sol.



$\frac{2\cos\theta\times x}{4} + y\times\sin\theta = 1$

$x\cos\theta + 2y\sin\theta = 2$

For B

$x = -2$

$-2\cos\theta + 2y\sin\theta = 2$

$y\sin\theta = 1 + \cos\theta$

$y = \dfrac{1+\cos\theta}{\sin\theta}$

For C

$x = 2$

$2\cos\theta + 2y\sin\theta = 2$

$y\sin\theta = 1 - \cos\theta$

$y = \dfrac{1-\cos\theta}{\sin\theta}$

$B\left(-2, \cot\theta/2\right)\ C\left(2, \tan\theta/2\right)$

$(x+2)(x-2) + \left(y - \cot\theta/2\right)\left(y - \tan\theta/2\right) = 0$

$x^2 - 4 + y^2 - y\left(\tan\theta/2 + \cot\theta/2\right) + 1 = 0$

$x^2 + y^2 - y\left(\tan\theta/2 + \cot\theta/2\right) - 3 = 0$

$\left(\sqrt{3}, 0\right) =$ Satisfies

12. If $P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then P^{50} is :

(1) $\begin{bmatrix} 1 & 0 \\ 25 & 1 \end{bmatrix}$

(2) $\begin{bmatrix} 1 & 50 \\ 0 & 1 \end{bmatrix}$

(3) $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 25 \\ 0 & 1 \end{bmatrix}$

Ans. Official Answer NTA (1)

Sol. $P = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$P^3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix}$

$P^4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$P^4 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$P^{50} = \begin{bmatrix} 1 & 0 \\ 25 & 1 \end{bmatrix}$

13. Consider the statement "The match will be played only if the weather is good and ground is not wet". Select the correct negation from the following :

(1) The match will not be played or weather is good and ground is not wet.

(2) The match will be played and weather is not good or ground is wet.

(3) The match will not be played and weather is not good and ground is wet.

(4) If the match will not be played, then either weather is not good or ground is wet.

Ans. Official Answer NTA (2)

Sol. P: Weather is good

q: ground is not wet

$\sim(p \vee q) \equiv \sim p v \sim q$

The match will be played and weather is not good or ground is wet.

14. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is :

(1) 1

(2) 4

(3) 2

(4) 3

Ans. Official Answer NTA (1)

Sol. $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

$C_1 \to C_1 - C_2 \quad C_2 \to C_2 - C_3$

$$\begin{vmatrix} \sin x - \cos x & 0 & \cos x \\ -(\sin x - \cos x) & \sin x - \cos x & \cos x = 0 \\ 0 & -(\sin x - \cos x) & \sin x \end{vmatrix}$$

$$(\sin x - \cos x)^2 \begin{vmatrix} 1 & 0 & \cos x \\ -1 & 1 & \cos x \\ 0 & -1 & \sin x \end{vmatrix} = 0$$

$(\sin x - \cos x)^2 \left[1(\sin x + \cos x) + \cos x(1)\right] = 0$

$\sin x = \cos x \qquad \sin x + 2\cos x = 0$

$\tan x = 1 \qquad \tan x = -2$

$x = \pi/4$

15. Let the equation of the pair of lines, $y = px$ and $y = qx$, can be written as $(y - px)(y - qx) = 0$. Then the equation of the pair of the angle bisectors of the lines $x^2 - 4xy - 5y^2 = 0$ is :

(1) $x^2 - 3xy + y^2 = 0$

(2) $x^2 + 3xy - y^2 = 0$

(3) $x^2 + 4xy - y^2 = 0$

(4) $x^2 - 3xy - y^2 = 0$

Ans. Official Answer NTA(2)

Sol. $x^2 - 4xy - 5y^2 = 0$

$\frac{x^2 - y^2}{1-(-5)} = \frac{xy}{-2}$

$\frac{x^2 - y^2}{6} = \frac{xy}{-2}$

$-x^2 + y^2 = 3xy$

$x^2 - y^2 + 3xy = 0$

16. If [x] be the greatest integer less than or equal to x, then $\sum_{n=8}^{100}\left[\frac{(-1)^n n}{2}\right]$ is equal to :

(1) 0

(2) 4

(3) 2

(4) -2

Ans. Official Answer NTA (2)

Sol. $\sum_{n=8}^{100}\left[\frac{(-1)^n n}{2}\right]$

$4-5+5-6+6........50+50=4$

17. The number of real solutions of the equation, $x^2 - |x| - 12 = 0$ is :

(1) 4

(2) 3

(3) 1

(4) 2

Ans. Official Answer NTA (4)

Sol. $x^2 - |x| - 12 = 0$

$t^2 - t - 12 = 0$

$(t-4)(t+3) = 0 = t = 4, -3$

$|x| = 4 \quad |x| = -3$

$x = \pm 4$

18. The sum of all those terms which are rational numbers in the expansion of $\left(2^{1/3}+3^{1/4}\right)^{12}$ is :

(1) 43

(2) 89

(3) 35

(4) 27

Ans. Official Answer NTA (1)

Sol. $\left(2^{1/3}+3^{1/4}\right)^{12}$

$T_{r+1}={}^{12}C_r\left(2^{1/3}\right)^{12-r}\left(3^{1/4}\right)^{r}$

$={}^{12}C_r\times 2^{\frac{12-r}{3}}\times 3^{r/4}$

$\frac{12-r}{3}=I \qquad \frac{r}{4}=I$

$r=0,3,6,9,12 \qquad r=0,4,8,12$

$r=0,12$

$T_1={}^{12}C_0\,2^4\times 1=16$

$T_{13}={}^{12}C_{12}\times 1\times 3^3=27$

Sum $=16+27=43$

19. Let $y=y(x)$ be the solution of the differential equation $xdy=(y+x^3\cos x)\,dx$ with $y(\pi)=0$, then $y\left(\frac{\pi}{2}\right)$ is equla to :

(1) $\frac{\pi^2}{2}-\frac{\pi}{4}$

(2) $\frac{\pi^2}{4}-\frac{\pi}{2}$

(3) $\frac{\pi^2}{4}+\frac{\pi}{2}$

(4) $\frac{\pi^2}{2}+\frac{\pi}{4}$

Ans. Official Answer NTA(3)

Sol. $xdy = ydx + x^3 \cos x \, dy$

$\frac{xdy - ydx}{x^2} = x \cos x \, dx$

$\int d(y/x) = \int x \cos x \, dx$

$\frac{y}{x} = x \int \cos x \, dx - \int \left(dx \int \cos x \, dx \right) dx$

$\frac{y}{x} = x \sin x - \int \sin x \, dx$

$\frac{y}{x} = x \sin x + \cos x + C$

$\frac{0}{\pi} = \pi \sin \pi + \cos \pi + C$

$0 = 0 - 1 + C$

$C = 1$

$\frac{y}{x} = x \sin x + \cos x + 1$

$\frac{y \times 2}{\pi} = \frac{\pi}{2} \sin \pi/2 + \cos \pi/2 + 1$

$y \times \frac{2}{\pi} = \frac{\pi}{2} + 1$

$y = \frac{\pi^2}{4} + \frac{\pi}{2}$

20. Consider functions $f : A \to B$ and $g : B \to C \, (A, B, C \subseteq R)$ such that $(gof)^{-1}$ exists, then :

(1) f and g both are one-one

(2) f is onto and g is one-one

(3) f and g both are onto

(4) f is one-one and g is onto

Ans. Official Answer NTA(4)

Sol. $(gof)^{-1}$ exists $\Rightarrow$ gof is bijective

f(x) should be one-one

g(x) should be onto

SECTION - B

1. If the co-efficients of x^7 and x^8 in the expansion of $\left(2+\frac{x}{3}\right)^n$ are equal, then the value of n is equal to ________.

Ans. Official Answer NTA (55)

Sol. $\left(2+\frac{x}{3}\right)^n$

$$T_{r+1} = {}^nC_r(2)^{n-r}\left(\frac{x}{3}\right)^r$$

$$r = 7 \quad r = 8$$

$$T_8 = {}^nC_7(2)^{n-7}(3)^{-7}(x)^7$$

$$T_9 = {}^nC_8(2)^{n-8}(3)^{-8}(x)^8$$

$${}^nC_7(2)^{n-7}\ (3)^{-7} = {}^nC_8(2)^{n-8}(3)^{-8}$$

$$\frac{{}^nC_7 \times 2^{n-7}}{3^7} = \frac{{}^nC_8 \times 2^{n-8}}{3^8}$$

$$6 \times \frac{\lfloor n}{\lfloor 7 \lfloor n-7} = \frac{\lfloor n}{\lfloor 8 \lfloor n-8}$$

$$\frac{6}{n-7} = \frac{1}{8}$$

$$48 + 7 = n = n = 55$$

2. If the line $\frac{x-k}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x+1}{3} = \frac{y+2}{2} = \frac{z+3}{1}$ are co-planar, then the value of k is ____________ .

Ans. Official Answer NTA (1)

Sol. $\vec{r} = (k, 2, 3) + \lambda(1, 2, 3)$

$$\vec{r} = (-1, -2, -3) + \mu(3, 2, 1)$$

$$\vec{d} = (k+1)\,\hat{i} + 4\,\hat{j} + 6\hat{k}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ k+1 & 4 & 6 \end{vmatrix} = 0$$

$1(8)-2(18-k-1)+3(12-2k-2)=0$

$2k+10-6K-6=0$

$4=4k$

$k=1$

3. A fair coin is tossed n-times such that the probability of getting at least one head is at leat 0.9. Then the minimum value of n is ___________.

Ans. Official Answer NTA (4)

Sol. $P(\text{Head})=\frac{1}{2}$

$1-P(\text{all tail})\geq 0.9$

$1-\frac{1}{2^n}\geq 0.9$

$0.1\geq\frac{1}{2^n}$

$\frac{1}{10}\geq\frac{1}{2^n}$

$\frac{1}{2^n}\leq\frac{1}{10}$

$2^n\geq 10$

$n=4$

4. Let a curve $y=f(x)$ pass through the point $(2,(\log_e 2)^2)$ and have slope $\frac{2y}{x\log_e x}$ for all positive real value of x. Then the value of f(e) is equal to ________.

Ans. Official Answer NTA (1)

Sol. $\frac{dy}{dx}=\frac{2y}{x\ln x}$

$\frac{dy}{2y}=\frac{dx}{x\ln x}$

$\frac{1}{2}\int\frac{dy}{y}=\int\frac{dx}{x\ln x}$

$\frac{1}{2}\ln y=\int\frac{dx}{x\ln x}$

$\ln x=t$

$$\frac{dx}{x} = dt$$

$$\frac{\ln y}{2} = \int \frac{dt}{t} \Rightarrow \frac{\ln y}{2} = \ln t + c$$

$$\frac{\ln y}{2} = \ln(\ln x) + c$$

$$\frac{\ln(\ln 2)^2}{2} = \ln(\ln 2) + c$$

$$\frac{2\ln(\ln 2)}{2} = \ln(\ln 2) + c$$

$$C = 0$$

$$\frac{\ln y}{2} = \ln(\ln x)$$

$$y = e^{2\ln(\ln x)}$$

$$f(x) = e^{2\ln(\ln x)}$$

$$f(e) = e^{2\ln(\ln e)} = 1$$

5. Let $n \in N$ and [x] denote the greatest integer less than or equal to x. If the sum of $(n+1)$ terms ${}^nC_0, 3.{}^nC_1, 5.{}^nC_2, 7.{}^nC_3, ...$ is equal to $2^{100}.101$, then $2\left[\frac{n-1}{2}\right]$ is equal to ___________.

Ans. Official Answer NTA (98)

Sol. $S = {}^nC_0 + 3\,{}^nC_1 + 5\,{}^nC_2 + ----- + (2n+1)\,{}^nC_n$

$$S = (2n+1)\,{}^nC_n + (2n-1)\,{}^nC_{n-1} + ----- + {}^nC_0$$

$$S = (2n+1)\,{}^nC_0 + (2n-1)\,{}^nC_1 + ------ + {}^nC_n$$

$$2S = (2n+2)\left({}^nC_0 + {}^nC_1 + ----- + {}^nC_n\right)$$

$$2S = 2(n+1) \times 2^n$$

$$S = (n+1)2^n = 2^{100} \times 101$$

$$n = 100$$

$$2\left[\frac{n-1}{2}\right] = 2\left[\frac{100-1}{2}\right] = 2\left[\frac{99}{2}\right] = 98$$

6. If $\left(\vec{a}+3\vec{b}\right)$ is perpendicular to $\left(7\vec{a}-5\vec{b}\right)$ and $\left(\vec{a}-4\vec{b}\right)$ is perpendicular to $\left(7\vec{a}-2\vec{b}\right)$, then the angle between $\vec{a}$ and $\vec{b}$ (in degrees) is __________.

Ans. Official Answer NTA (60)

Sol. $\left(\vec{a}+3\vec{b}\right).\left(7\vec{a}-5\vec{b}\right)=0$

$7|\vec{a}|^2-5\vec{a}\cdot\vec{b}+21\vec{a}\cdot\vec{b}-15|\vec{b}|^2=0$

$7|\vec{a}|^2+16\vec{a}\cdot\vec{b}-15|\vec{b}|^2=0---\{1\}$

$(\vec{a}-4\vec{b}).(7\vec{a}-2\vec{b})=0$

$7|\vec{a}|^2-2\vec{a}\cdot\vec{b}-28\ \vec{a}\cdot\vec{b}+8|\vec{b}|^2=0$

$7|\vec{a}|^2-30\ \vec{a}\cdot\vec{b}+8|\vec{b}|^2=0----(2)$

$46\vec{a}\cdot\vec{b}-23|\vec{b}|^2=0$

$\Rightarrow \vec{a}\cdot\vec{b}=\frac{|\vec{b}|^2}{2}$

$7|\vec{a}|^2+16\vec{a}\cdot\vec{b}-30\ \vec{a}\cdot\vec{b}=0$

$7|\vec{a}|^2=14\vec{a}\cdot\vec{b}$

$|\vec{a}|^2=2\vec{a}\cdot\vec{b} \quad \Rightarrow \quad \vec{a}\cdot\vec{b}=\frac{|\vec{a}|^2}{2}$

$\frac{|\vec{b}|^2}{2}=\frac{|\vec{a}|^2}{2}\Rightarrow|\vec{a}|=|\vec{b}|$

$2\vec{a}\cdot\vec{b}=|\vec{b}|^2$

$2|\vec{a}||\vec{b}|\cos\theta=|\vec{b}|^2$

$\cos\theta=1/2\Rightarrow\theta=\pi/3$

7. Consider the function $f(x)=\frac{P(x)}{\sin(x-2)}, \quad x\neq 2$

$\qquad =7 \qquad , x=2$

where P(x) is a polynomial such that P"(x) is always a constant and P(3) = 9. If f(x) is continuous at x = 2, then P(5) is equal to ____________.

Ans. Official Answer NTA (39)

Sol. $f(x)=\begin{cases}\dfrac{P(x)}{\sin(x-2)} & x\neq 2\\ 7 & x=2\end{cases}$

$P''(x)=\text{constant} \quad \Rightarrow P(x)$ is 2 degree polynomial

$f(2^{+})=f(2^{-})=f(2)$

$\lim\limits_{x\to 2}\dfrac{P(x)}{\sin(x-2)}=7$

$\lim\limits_{x\to 2}\dfrac{(x-2)P(x)}{(x-2)\sin(x-2)}=7$

$\lim\limits_{x\to 2}\dfrac{P(x)}{(x-2)}=7$

$P(x)=(x-2)(ax+b)$

$2a+b=7$

$P(3)=3a+b=9$

$a=2$

$4+b=7\Rightarrow b=3$

$P(x)=(x-2)(2x+3)$

$P(5)=3(10+3)=39$

8. The equation of a circle is $\text{Re}(z^2)+2(\text{Im}(z))^2+2\,\text{Re}(z)=0$, where $z=x+iy$. A line which passes through the center of the given circle and the vertex of the parabola, $x^2-6x-y+13=0$, has y-intercept equal to ____________.

Ans. Official Answer NTA (1)

Sol. $Z=x+iy$

$Z^2=(x+iy)^2=x^2-y^2+2ixy$

$\text{Re}(Z^2)=x^2-y^2 \quad \text{Im}(z)=y$

$x^2-y^2+2y^2+2x=0$

$x^2+y^2+2x=0$

centre $(-1,0)$

$x^2-6x=y-13$

$(x-3)^2-9=y-13$

$(x-3)^2 = y-4$

$x-3 = X \quad y-4 = Y$

$X^2 = Y$

Vertex = (3, 4)

$\frac{4-0}{4} = \frac{y-0}{x+1}$

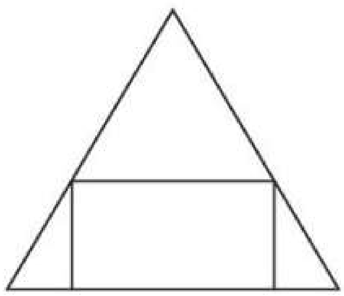
$1 = \frac{y}{x+1}$

$n+1 = y$

Put $x = 0$

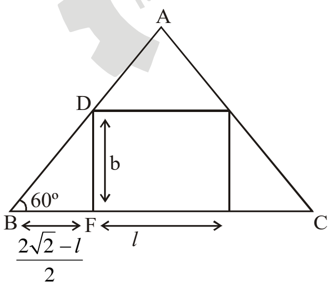
yint = 1

9. If a rectangle is inscribed in an equilateral triangle of side length $2\sqrt{2}$ as shown in the figure, then the square of the largest area of such a rectangle is _______.



Ans. Official Answer NTA (3)

Sol.



In D DBF

$\tan 60^\circ = \frac{2b}{2\sqrt{2}-l} \Rightarrow b = \frac{\sqrt{3}(2\sqrt{2}-l)}{2}$

$A = l \times b = l \times \frac{\sqrt{3}}{2}(2\sqrt{2}-l)$

$$\frac{dA}{dl}=\frac{\sqrt{3}}{2}\left(2\sqrt{2}-2l\right)=0$$

$$\sqrt{3}\left(\sqrt{2}-l\right)=0$$

$$l=\sqrt{2}$$

$$A=l\times b=\sqrt{2}\left(\frac{\sqrt{3}}{2}\right)\left(2\sqrt{2}-\sqrt{2}\right)$$

$$=\frac{\sqrt{3}}{2}(2)=\sqrt{3}$$

$A^2 = 3$

10. If $a+b+c=1, ab+bc+ca=2$ and $abc=3$, then the value of $a^4+b^4+c^4$ is equal to _________.

Ans. Official Answer NTA (13)

Sol. $a + b + c = 1$ $\quad$ $ab + bc + ca = 2$ $\quad$ $abc = 3$

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$1 = a^2+b^2+c^2 + 4$

$a^2+b^2+c^2 = -3$

$(a^2+b^2+c^2)^2 = a^4+b^4+c^4 + 2(a^2b^2+b^2c^2+c^2a^2)$

$9-2(a^2b^2+b^2c^2+c^2a^2) = a^4 + b^4 + c^4$

$(ab+bc+ca)^2 = a^2b^2+b^2c^2+c^2a^2 + 2(ab^2c + abc^2 + a^2bc)$

$4 = a^2b^2+b^2c^2+c^2a^2+ 2abc(b+c+a)$

$4 = a^2b^2+b^2c^2+c^2a^2+ 6$

$-2 = a^2b^2+b^2c^2+c^2a^2$

$a^4 + b^4 + c^4 = 9 - 2(-2)$

$a^4+b^4+c^4 = 13$