JEE Main July 2022

Question Paper With Text Solution

25 July | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JULY 2022 | 25TH JULY SHIFT-2

SECTION – A

Question ID : 156941

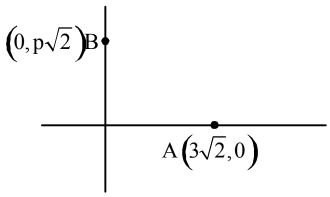
Complex Number

1. For $z \in C$ if the minimum value of $\left(\left|z-3\sqrt{2}\right|+\left|z-p\sqrt{2}i\right|\right)$ is $5\sqrt{2}$, then a value of p is :

$z \in C$, के लिए यदि $\left(\left|z-3\sqrt{2}\right|+\left|z-p\sqrt{2}i\right|\right)$ का निम्नतम मान $5\sqrt{2}$ है, तो p का एक मान है :

(1) 3 (2) $\frac{7}{2}$ (3) 4 (4) $\frac{9}{2}$

Ans. Official Answer NTA (3)

Sol. 

It is sum of distance of *z* from $\left(3\sqrt{2},0\right)$ and $\left(0,p\sqrt{2}\right)$

For minimising, z should lie on AB and AB = $5\sqrt{2}$

$(AB)^2 = 18 + 2p^2$

$p = \pm 4$

Question ID : 156942

Determinant

2. The number of real values of λ, such that the system of linear equations

$2x - 3y + 5z = 9$

$x + 3y - z = -18$

$3x - y + (\lambda^2 - |\lambda|)z = 16$

has no solutions, is :

λ के वास्तविक मानों, जिनके लिए रैखिक समीकरण निकाय

$2x - 3y + 5z = 9$

$x + 3y - z = -18$

$3x - y + (\lambda^2 - |\lambda|)z = 16$

का कोई हल नहीं है, की संख्या है :

(1) 0 (2) 1 (3) 2 (4) 4

Ans. Official Answer NTA (3)

Sol. $\Delta = \begin{vmatrix} 2 & -3 & 5 \\ 1 & 3 & -1 \\ 3 & -1 & \lambda^2 - |\lambda| \end{vmatrix} = 2\left(3\lambda^2 - 3|\lambda| - 1\right)$

$+3\left(\lambda^2 - |\lambda| + 3\right)$

$+5(-1 - 9)$

$= 9\lambda^2 - 9|\lambda| - 43$

$= 9|\lambda|^2 - 9|\lambda| - 43$

$\Delta = 0$ for 2 values of $|\lambda|$ out of which one is –ve and other is +ve

So, 2 values of λ satisfy the system of equations to obtain no solution

Question ID : 156943

Function

3. The number of bijective functions $f : \{1, 3, 5, 7, ..., 99\} \rightarrow \{2, 4, 6, 8, ..., 100\}$, such that $f(3) \geq f(9) \geq f(15) \geq f(21) \geq .... \geq f(99)$, is :

एकैकी आच्छादी फलनों $f : \{1, 3, 5, 7, ..., 99\} \rightarrow \{2, 4, 6, 8, ..., 100\}$ जिनके लिए $f(3) \geq f(9) \geq f(15) \geq f(21) \geq .... \geq f(99)$, हैं, की संख्या है :

(1) ${}^{50}P_{17}$ (2) ${}^{50}P_{33}$ (3) $33! \times 17!$ (4) $\frac{50!}{2}$

Ans. Official Answer NTA (2)

Sol. Let number of terms in sequence

$\Rightarrow 3, 9, 15, ..., 99$ be n

$\Rightarrow 99 = 3 + (n - 1)\,6 \Rightarrow n = 17$

number of bijective functions $= {}^{50}C_{17}.33!$

$= {}^{50}C_{33}.33! = {}^{50}P_{33}$

Question ID : 156944

Binomial Theorem

4. The remainder when $(11)^{1011} + (1011)^{11}$ is divided by 9 is :

$(11)^{1011} + (1011)^{11}$ को 9 से विभाजित करने पर शेषफल है :

(1) 1 (2) 4 (3) 6 (4) 8

Ans. Official Answer NTA (4)

Sol. $\text{Re}\left(\frac{(11)^{1011}+(1011)^{11}}{9}\right)=\text{Re}\left(\frac{2^{1011}+3^{11}}{9}\right)$

For $\text{Re}\left(\frac{2^{1011}}{9}\right)$

$2^{1011}=(9-1)^{337}={}^{337}C_0 9^{337}(-1)^0+{}^{337}C_1 9^{336}(-1)^1+{}^{337}C_2 9^{335}(-1)^2+\ldots..+{}^{337}C_{337}9^0(-1)^{337}$

so, remainder is 8

and $\text{Re}\left(\frac{3^{11}}{9}\right)=0$

so, remainder is 8

Question ID : 156945

Sequence & progression

5. The sum $\sum_{n=1}^{21}\frac{3}{(4n-1)(4n+3)}$ is equal to :

योगफल $\sum_{n=1}^{21}\frac{3}{(4n-1)(4n+3)}$ बराबर है :

(1) $\frac{7}{87}$ (2) $\frac{7}{29}$ (3) $\frac{14}{87}$ (4) $\frac{21}{29}$

Ans. Official Answer NTA (2)

Sol. $\sum_{n=1}^{21}\frac{3}{(4n-1)(4n+3)}=\frac{3}{4}\sum_{n=1}^{21}\frac{1}{4n-1}-\frac{1}{4n+3}$

$=\frac{3}{4}\left[\left(\frac{1}{3}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{11}\right)+\ldots\ldots+\left(\frac{1}{83}-\frac{1}{87}\right)\right]$

$=\frac{3}{4}\left[\frac{1}{3}-\frac{1}{87}\right]=\frac{3}{4}\frac{84}{3.87}=\frac{7}{29}$

Question ID : 156946

Limit

6. $\lim_{x\to\frac{\pi}{4}}\frac{8\sqrt{2}-(\cos x+\sin x)^7}{\sqrt{2}-\sqrt{2}\sin 2x}$ is equal to :

$\lim\limits_{x\to\frac{\pi}{4}}\frac{8\sqrt{2}-(\cos x+\sin x)^7}{\sqrt{2}-\sqrt{2}\sin 2x}$ बराबर है :

(1) 14 (2) 7 (3) $14\sqrt{2}$ (4) $7\sqrt{2}$

Ans. Official Answer NTA (1)

Sol. $\lim\limits_{x\to\frac{\pi}{4}}\frac{8\sqrt{2}-(\sqrt{2})^7\left(\cos\left(x-\frac{\pi}{4}\right)\right)^7}{\sqrt{2}(1-\sin 2x)}$

$x=\frac{\pi}{4}+h$

$\lim\limits_{h\to 0}\frac{8\sqrt{2}(1-\cos^7 h)}{\sqrt{2}(1-\cos 2h)}$

L–Hospital

$\lim\limits_{h\to 0}8\left(\frac{-7\cos^6 h(-\sinh)}{\sin 2h.2}\right)$

$\lim\limits_{h\to 0}\frac{8\times 7\sinh\cos^6 h}{4\sin\cosh}$

$=\lim\limits_{h\to 0}\frac{8\times 7}{4}\cos^5=14$

Question ID : 156947

Definite Integration

7. $\lim\limits_{n\to\infty}\frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}}+\frac{1}{\sqrt{1-\frac{2}{2^n}}}+\frac{1}{\sqrt{1-\frac{3}{2^n}}}+\ldots..+\frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$ is equal to :

$\lim\limits_{n\to\infty}\frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}}+\frac{1}{\sqrt{1-\frac{2}{2^n}}}+\frac{1}{\sqrt{1-\frac{3}{2^n}}}+\ldots..+\frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$ बराबर है :

(1) $\frac{1}{2}$ (2) 1 (3) 2 (4) –2

Ans. Official Answer NTA (3)

Sol. $I = \lim_{n\to\infty} \frac{1}{2^n}\left(\frac{1}{\sqrt{1-\frac{1}{2^n}}}+\frac{1}{\sqrt{1-\frac{2}{2^n}}}+\frac{1}{\sqrt{1-\frac{3}{2^n}}}+\ldots..+\frac{1}{\sqrt{1-\frac{2^n-1}{2^n}}}\right)$

Let $2^n = t$ and if $n\to\infty$ then $t\to\infty$

$I = \lim_{n\to\infty}\frac{1}{t}\left(\sum_{r=1}^{t-1}\frac{1}{\sqrt{1-\frac{r}{t}}}\right)$

$I = \int_0^1 \frac{dx}{\sqrt{1-x}} = \int_0^1 \frac{dx}{\sqrt{x}} \quad \left(\int_0^0 f(x)dx = \int_0^0 f(a-x)dx\right.$

$= \left[2x^{\frac{1}{2}}\right]_0^1 = 2$

Question ID : 156948

Probability

8. If A and B are two events such that $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{5}$ and $P(A\cup B) = \frac{1}{2}$, then $P(A|B') + P(B|A')$ is equal to :

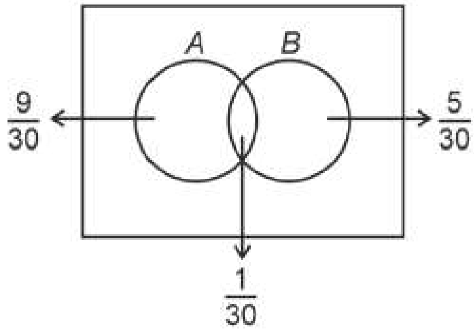
यदि दो घटनाओं A तथा B के लिए $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{5}$ तथा $P(A\cup B) = \frac{1}{2}$ हैं, तो $P(A|B') + P(B|A')$ बराबर है :

(1) $\frac{3}{4}$ (2) $\frac{5}{8}$ (3) $\frac{5}{4}$ (4) $\frac{7}{8}$

Ans. Official Answer NTA (2)

Sol. $P(A) = \frac{1}{3}, P(B) = \frac{1}{5}$ and $P(A\cup B) = \frac{1}{2}$

$\therefore\ P(A\cap B) = \frac{1}{3}+\frac{1}{5}-\frac{1}{2} = \frac{1}{30}$



Now, $P(A|B') + P(B|A') = \frac{P(A \cap B')}{P(B')} + \frac{P(B \cap A')}{P(A')}$

$$= \frac{\frac{9}{30}}{\frac{4}{5}} + \frac{\frac{5}{30}}{\frac{2}{3}} = \frac{5}{8}$$

Question ID : 156949

Definite Integration

9. Let [t] denote the greatest integer less than or equal to t. Then the value of the integral $\int_{-3}^{101}\left(\left[\sin(\pi x)\right] + e^{\left[\cos(2\pi x)\right]}\right)dx$ is equal to :

माना कि [t] वो महत्तम पूर्णांक है जो t से कम या बराबर है। तो समाकलन $\int_{-3}^{101}\left(\left[\sin(\pi x)\right] + e^{\left[\cos(2\pi x)\right]}\right)dx$ का मान बराबर है :

(1) $\frac{52(1-e)}{e}$ (2) $\frac{52}{e}$ (3) $\frac{52(2+e)}{e}$ (4) $\frac{104}{e}$

Ans. Official Answer NTA (2)

Sol. $I = \int_{-3}^{101}\left([\sin(\pi x)] + e^{[\cos(2\pi x)]}\right)dx$

$[\sin \pi x]$ is periodic with period 2 and $e^{[\cos(2\pi x)]}$ is periodic with period 1.

So,

$$I = 52\int_0^2 \left([\sin \pi x] + e^{[\cos 2\pi x]}\right)dx$$

$$= 52\left\{\int_1^2 -1dx + \int_{\frac{1}{4}}^{\frac{3}{4}} e^{-1}dx + \int_{\frac{5}{4}}^{\frac{7}{4}} e^{-1}dx + \int_0^{\frac{1}{4}} e^0 dx + \int_{\frac{3}{4}}^{\frac{5}{4}} e^0 dx + \int_{\frac{7}{4}}^{2} e^0 dx\right\}$$

$$= \frac{52}{e}$$

Question ID : 1569410

Straight Line

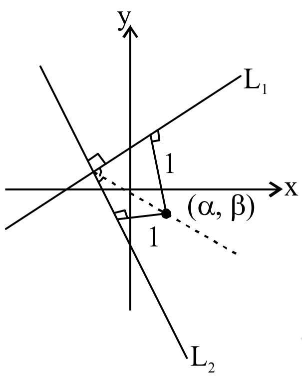
10. Let the point $P(\alpha, \beta)$ be at a unit distance from each of the two lines $L_1 : 3x - 4y + 12 = 0$, and $L_2 : 8x + 6y + 11 = 0$. If P lies below L_1 and above L_2, then $100(\alpha + \beta)$ is equal to :

माना बिन्दु $P(\alpha, \beta)$ दो रेखाओं $L_1 : 3x - 4y + 12 = 0$, तथा $L_2 : 8x + 6y + 11 = 0$ में से प्रत्येक से इकाई दूरी पर है। यदि P, L_1 के नीचे तथा L_2 के ऊपर स्थित है, तो $100(\alpha + \beta)$ बराबर है :

(1) –14 (2) 42 (3) –22 (4) 14

Ans. Official Answer NTA (4)

Sol.



L_1: 3x - 4y + 12 = 0

L_2: 8x + 6y + 11 = 0

Equation of angle bisector of L_1 and L_2 of angle containing origin

$2(3x - 4y + 12) = 8x + 6y + 11$

$2x + 14y - 13 = 0$ __________(i)

$\frac{3\alpha - 4\beta + 12}{5} = 1$

$\Rightarrow 3\alpha - 4\beta + 7 = 0$ __________(ii)

Solution of $2x + 14y - 13 = 0$ and $3x - 4y + 7 = 0$ gives the required point $P(\alpha, \beta), \alpha = \frac{-23}{25}, \beta = \frac{53}{50}$

$100(\alpha + \beta) = 14$

Question ID : 1569411

Differential Equation

11. Let a smooth curve $y = f(x)$ be such that the slope of the tangent at any point (x, y) on it is directly proportional to $\left(\frac{-y}{x}\right)$. If the curve passes through the points (1, 2) and (8, 1), then $\left|y\left(\frac{1}{8}\right)\right|$ is equal to :

माना एक निष्कोण वक्र $y = f(x)$ के किसी भी बिन्दु (x, y) पर स्पर्श रेखा की प्रवणता $\left(\frac{-y}{x}\right)$ के अनुक्रमानुपात में है। यदि यह वक्र बिन्दुओं (1, 2) तथा (8, 1) से होकर जाता है, तो $\left|y\left(\frac{1}{8}\right)\right|$ बराबर है :

(1) $2\log_e 2$ (2) 4 (3) 1 (4) $4\log_e 2$

Ans. Official Answer NTA (2)

Sol. $\frac{dy}{dx} \propto \frac{-y}{x}$

$\frac{dy}{dx} = \frac{-ky}{x} \Rightarrow \int \frac{dy}{y} = -K \int \frac{dx}{x}$

$\ln|y| = -K \ln|x| + C$

If the above equation staisfy (1, 2) and (8, 1)

$\ln 2 = -K \times 0 + C \Rightarrow C = \ln 2$

$\ln 1 = -K \ln 8 + \ln 2 \Rightarrow K = \frac{1}{3}$

So, at $x = \frac{1}{8}$

$\ln|y| = -\frac{1}{3}\ln\left(\frac{1}{8}\right) + \ln 2 = 2\ln 2$

$|y| = 4$

Question ID : 1569412

Ellipse

12. If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the line $\frac{x}{7} + \frac{y}{2\sqrt{6}} = 1$ on the x-axis and the line $\frac{x}{7} - \frac{y}{2\sqrt{6}} = 1$ on the y-axis, then the eccentricity of the ellipse is :

यदि दीर्घवृत्त $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, रेखा $\frac{x}{7} + \frac{y}{2\sqrt{6}} = 1$ को x-अक्ष पर तथा रेखा $\frac{x}{7} - \frac{y}{2\sqrt{6}} = 1$ को y-अक्ष पर मिलता है, तो दीर्घवृत्त की उत्केन्द्रता है :

(1) $\frac{5}{7}$ (2) $\frac{2\sqrt{6}}{7}$ (3) $\frac{3}{7}$ (4) $\frac{2\sqrt{5}}{7}$

Ans. Official Answer NTA (1)

Sol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the line $\frac{x}{7}+\frac{y}{2\sqrt{6}}=1$ on the x-axis

So, a = 7 and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet the line $\frac{x}{7}-\frac{y}{2\sqrt{6}}=1$ y-axis

So, b = $2\sqrt{6}$

Therefore, $e^2=1-\frac{b^2}{a^2}=1-\frac{24}{49}$

$e=\frac{5}{7}$

Question ID : 1569413

Area Under Curve

13. The tangents at the points A(1, 3) and B(1, –1) on the parabola $y^2-2x-2y=1$ meet at the point P. Then the area (in unit^2) of the triangle PAB is :

माना परवलय $y^2-2x-2y=1$ के बिन्दुओं A(1, 3) तथा B(1, –1) पर स्पर्श रेखाएँ बिन्दु P पर मिलती हैं। तो त्रिभुज PAB का क्षेत्रफल (वर्ग इकाई में) है :

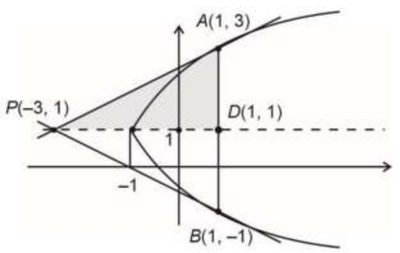
(1) 4 (2) 6 (3) 7 (4) 8

Ans. Official Answer NTA (4)

Sol. Given curve : y2 – 2x – 2y = 1.

Can be written as

$(y-1)^2=2(x+1)$



And, the given information

Can be plotted as shown in figure

Tangent at A : 2y – x – 5 = 0 {using T = 0}

Intersection with y = 1 is x = –3

Hence, point P is (–3, 1)

Taking advantage of symmetry

Area of $\Delta PAB = 2\times\frac{1}{2}\times\left(1-\left(-3\right)\right)\times\left(3-1\right)$

= 8 sq. units

Question ID : 1569414

Hyperbola

14. Let the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{7}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{\alpha}=\frac{1}{25}$ coincide. Then the length of the latus rectum of the hyperbola is :

माना दीर्घवृत्त $\frac{x^2}{16}+\frac{y^2}{7}=1$ तथा अतिपरवलय $\frac{x^2}{144}-\frac{y^2}{\alpha}=\frac{1}{25}$ की नाभियाँ सम्पाती हैं। तो अतिपरवलय की नाभिलंब जीवा की लंबाई है :

(1) $\frac{32}{9}$ (2) $\frac{18}{5}$ (3) $\frac{27}{4}$ (4) $\frac{27}{10}$

Ans. Official Answer NTA (4)

Sol. Ellipse : $\frac{x^2}{16}+\frac{y^2}{7}=1$

Eccentricity $=\sqrt{1-\frac{7}{16}}=\frac{3}{4}$

Foci $\equiv\left(\pm a\,e,0\right)\equiv\left(\pm 3,0\right)$

Hyperbola : $\frac{x^2}{\left(\frac{144}{25}\right)}-\frac{y^2}{\left(\frac{\alpha}{25}\right)}=1$

Eccentricity $=\sqrt{1+\frac{\alpha}{144}}=\frac{1}{12}\sqrt{144+\alpha}$

Foci $\equiv\left(\pm ae,0\right)\equiv\left(\pm\frac{12}{5}.\frac{1}{12}\sqrt{144+\alpha},0\right)$

If foci coincide then $3=\frac{1}{5}\sqrt{144+\alpha}\Rightarrow\alpha=81$

Hence, hyperbola is $\frac{x^2}{\left(\frac{12}{5}\right)^2}-\frac{y^2}{\left(\frac{9}{5}\right)^2}=1$

Length of latus rectum $=2.\frac{\frac{81}{25}}{\frac{12}{5}}=\frac{27}{10}$

Question ID : 1569415

3D Geometry

15. A plane E is perpendicular to the two planes $2x-2y+z=0$ and $x-y+2z=4$, and passes through the point $P(1,-1,1)$. If the distance of the plane E from the point $Q(a,a,2)$ is $3\sqrt{2}$, then $(PQ)^2$ is equal to :

एक समतल E दो समतलों $2x-2y+z=0$ तथा $x-y+2z=4$ के लंबवत है, तथा बिन्दु $P(1,-1,1)$ से होकर जाता है। समतल E की बिन्दु $Q(a,a,2)$ से दूरी $3\sqrt{2}$ है, तो $(PQ)^2$ बराबर है :

(1) 9 (2) 12 (3) 21 (4) 33

Ans. Official Answer NTA (3)

Sol. First plane, $P_1=2x-2y+z=0$, normal vector $\equiv \overline{n}_1=\langle 2,-2,1\rangle$

Second plane, $P_2\equiv x-y+2z=4$, normal vector $\equiv \overline{n}_2=(1,-1,2)$

Plane perpendicular to P_1 and P_2 will have normal vector $\overline{n}_3$

Where $\overline{n}_3=\left(\overline{n}_1\times\overline{n}_2\right)$

Hence, $\overline{n}_3=(-3,-3,0)$

Equation of plane E through $P(1,-1,1)$ and $\overline{n}_3$ as normal vector

$(x-1,y+1,z-1)\cdot(-3,-3,0)=0$

$\Rightarrow x+y=0\equiv E$

Distance of PQ(a, a, 2) from $E=\left|\frac{2a}{\sqrt{2}}\right|$

as given, $\left|\frac{2a}{\sqrt{2}}\right| = 3\sqrt{2} \Rightarrow a = \pm 3$

Hence, $Q \equiv (\pm 3, \pm 3, 2)$

Distance $7Q = \sqrt{21} \Rightarrow (PQ)^2 = 21$

Question ID : 1569416

3D Geometry

16. The shortest distance between the lines $\frac{x+7}{-6} = \frac{y-6}{7} = z$ and $\frac{7-x}{2} = y-2 = z-6$ is :

रेखाओं $\frac{x+7}{-6} = \frac{y-6}{7} = z$ तथा $\frac{7-x}{2} = y-2 = z-6$ के बीच न्यूनतम दूरी है :

(1) $2\sqrt{29}$ (2) 1 (3) $\sqrt{\frac{37}{29}}$ (4) $\frac{\sqrt{29}}{2}$

Ans. Official Answer NTA (1)

Sol. $L_1 : \frac{x+7}{-6} = \frac{y-6}{7} = \frac{z-0}{1}$

Any point on it $\vec{a}_1$ (–7, 6, 0)

and L_1 is parallel to $\vec{b}_1$ (–6, 7, 1)

$L_2 : \frac{x-7}{-2} = \frac{y-2}{1} = \frac{z-6}{1}$

Any point on it, $\vec{a}_2$ (7, 2, 6)

and L_2 is parallel to $\vec{b}_2$ (–2, 1, 1)

Shortest distance between L_1 and L_2

$$= \left|\frac{(\vec{a}_1 - \vec{a}_2)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right| = \left|\frac{(-14, 4, -6)\cdot(3, 2, 4)}{\sqrt{9+4+16}}\right|$$

$$= 2\sqrt{29}$$

Question ID : 1569417

Vectors

17. Let $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ and let $\vec{b}$ be a vector such that $\vec{a} \times \vec{b} = 2\hat{i} - \hat{k}$ and $\vec{a} \cdot \vec{b} = 3$. Then the projection of $\vec{b}$ on the vector $\vec{a} - \vec{b}$ is :

माना $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$ तथा एक सदिश $\vec{b}$ के लिए $\vec{a} \times \vec{b} = 2\hat{i} - \hat{k}$ तथा $\vec{a} \cdot \vec{b} = 3$ हैं, तो सदिश $\vec{b}$ का सदिश $\vec{a} - \vec{b}$ पर प्रक्षेप है :

(1) $\frac{2}{\sqrt{21}}$ (2) $2\sqrt{\frac{3}{7}}$ (3) $\frac{2}{3}\sqrt{\frac{7}{3}}$ (4) $\frac{2}{3}$

Ans. Official Answer NTA (1)

Sol. $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$

$\vec{a} \times \vec{b} = 2\hat{i} - \hat{k}$

$\vec{a} \cdot \vec{b} = 3$

$|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 \cdot |\vec{b}|^2$

$\Rightarrow 5 + 9 = 6|\vec{b}|^2$

$\Rightarrow |b|^2 = \frac{7}{3}$

$|\vec{a} - \vec{b}| = \sqrt{|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}} = \sqrt{\frac{7}{3}}$

Projection of $\vec{b}$ on $\vec{a} - \vec{b} = \frac{\vec{b}.(\vec{a} - \vec{b})}{|\vec{a} - \vec{b}|}$

$= \frac{\vec{b}.\vec{a} - |\vec{b}|^2}{|\vec{a} - \vec{b}|} = \frac{3 - \frac{7}{3}}{\sqrt{\frac{7}{3}}}$

$= \frac{2}{\sqrt{21}}$

Question ID : 1569418

Statistics

18. If the mean deviation about median for the numbers 3, 5, 7, 2k, 12, 16, 21, 24, arranged in the ascending order, is 6 then the median is :

यदि आरोही क्रम में लिखी संख्याओं 3, 5, 7, 2k, 12, 16, 21, 24 का माध्यिका के सापेक्ष माध्य विचलन 6 है, तो माध्यिका है :

(1) 11.5 (2) 10.5 (3) 12 (4) 11

Ans. Official Answer NTA (4)

Sol. Median $= \frac{2k+12}{2} = k+6$

$$\text{M.D.} = \frac{\Sigma|x_i - M|}{n} = 6$$

$$= \frac{(k+3)+(k+1)+(k-1)+(6-k)+(6-k)+(10-k)+(15-k)+(18-k)}{8}$$

$$\Rightarrow \frac{58-2k}{8} = 6$$

$$\Rightarrow k = 5$$

Median = 5 + 6 = 11

Question ID : 1569419

Trigonometric Ratio and Identities

19. $2\sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right)$ is equal to :

$2\sin\left(\frac{\pi}{22}\right)\sin\left(\frac{3\pi}{22}\right)\sin\left(\frac{5\pi}{22}\right)\sin\left(\frac{7\pi}{22}\right)\sin\left(\frac{9\pi}{22}\right)$ बराबर है :

(1) $\frac{3}{16}$ (2) $\frac{1}{16}$ (3) $\frac{1}{32}$ (4) $\frac{9}{32}$

Ans. Official Answer NTA (2)

Sol. $2\sin\frac{\pi}{22}\sin\frac{3\pi}{22}\sin\frac{5\pi}{22}\sin\frac{7\pi}{22}\sin\frac{9\pi}{22}$

$$= 2\sin\left(\frac{11\pi-10\pi}{22}\right)\sin\left(\frac{11\pi-8\pi}{22}\right)\sin\left(\frac{11\pi-6\pi}{22}\right)\sin\left(\frac{11\pi-4\pi}{22}\right)\sin\left(\frac{11\pi-2\pi}{22}\right)$$

$$= 2\cos\frac{\pi}{11}\cos\frac{2\pi}{11}\cos\frac{3\pi}{11}\cos\frac{4\pi}{11}\cos\frac{5\pi}{11}$$

$$= \frac{2\sin\frac{32\pi}{11}}{2^5\sin\frac{\pi}{11}}$$

$$= \frac{1}{16}$$

Question ID : 1569420

Mathematical Reasoning

20. Consider the following statements :

P : Ramu is intelligent.

Q : Ramu is rich.

R : Ramu is not honest

The negation of the statement "Ramu is intelligent and honest if and only if Ramu is not rich" can be expressed as :

निम्न कथनों का विचार कीजिए :

P : रामू बुद्धिमान है

Q : रामू धनी है

R : रामू ईमानदार नहीं है

कथन "रामू बुद्धिमान तथा ईमानदार है यदि और केवल यदि रामू धनी नहीं है" के निषेधन को किस से व्यक्त कर सकते हैं :

(1) $((P \wedge (\sim R)) \wedge Q) \wedge ((\sim Q) \wedge ((\sim P) \vee R))$

(2) $((P \wedge R) \wedge Q) \vee ((\sim Q) \wedge ((\sim P) \vee (\sim R)))$

(3) $((P \wedge R) \wedge Q) \wedge ((\sim Q) \wedge ((\sim P) \vee (\sim R)))$

(4) $((P \wedge (\sim R)) \wedge Q) \vee ((\sim Q) \wedge ((\sim P) \vee R))$

Ans. Official Answer NTA (4)

Sol. P : Ramu is intelligent

Q : Ramu is rich

R : Ramu is not honest

Given statement, "Ramu is intelligent and honest if and only if Ramu is not rich"

$= (P \wedge \sim R) \Leftrightarrow \sim Q$

So, negation of the statement is

$\sim [(P \wedge \sim R) \Leftrightarrow \sim Q]$

$= \sim [\{\sim (P \wedge \sim R) \vee \sim Q\} \wedge \{Q \vee (P \wedge \sim R)\}]$

$= ((P \wedge \sim R) \wedge Q) \vee (\sim Q \wedge (\sim P \vee R))$

SECTION – B

Question ID : 1569421

Set & Relations

21. Let A = {1, 2, 3, 4, 5, 6, 7}. Define B = {T ⊆ A : either $1 \notin T$ or $2 \in T$} and C = {T ⊆ A : T the sum of all the elements of T is a prime number}. Then the number of elements in the set $B \cup C$ is _________.

माना A = {1, 2, 3, 4, 5, 6, 7} है। समुच्चय B = {T ⊆ A : या तो $1 \notin T$ या $2 \in T$} तथा C = {T ⊆ A : T के सभी अवयवों का योगफल एक अभाज्य संख्या है} हैं। तो समुच्चय $B \cup C$ में अवयवों की संख्या है _________

Ans. Official Answer NTA (107)

Sol. $\because (B \cup C)' = B' \cap C'$

B' is a set containing sub sets of A containing element 1 and not containing 2.

And C' is a set containing subsets of A whose sum of elements is not prime.

So, we need to calculate number of subsets of

{3, 4, 5, 6, 7} whose sum of elements plus 1 is composite.

Number of such 5 elements subset = 1

Number of such 4 elements subset = 3 (except selecting 3 or 7)

Number of such 3 elements subset = 6 (except selecting {3, 4, 5}, {3, 6, 7}, {4, 5, 7} or {5, 6, 7})

Number of such 2 elements subset = 7 (except selecting {3, 7}, {4, 6}, {5, 7})

Number of such 1 elements subset = 3 (except selecting {4} or {6})

Number of such 0 elements subset = 1

$n(B' \cap C') = 21 \Rightarrow n(B \cup C) = 2^7 - 21 = 107$

Question ID : 1569422

Function

22. Let f(x) be a quadratic polynomial with leading coefficient 1 such that f(0) = p, p ≠ 0, and $f(1) = \frac{1}{3}$. If the equations f(x) = 0 and fofofof(x) = 0 have a common real root, then f(–3) is equal to _________.

माना f(x) एक द्विघाती बहुपद है जिसका अग्रग–गुणांक 1 है तथा f(0) = p, p ≠ 0 और $f(1) = \frac{1}{3}$ हैं। यदि समीकरणों f(x) = 0 तथा fofofof(x) = 0 का एक उभ्यनिष्ठ वास्तविक मूल है, तो f(–3) बराबर है _________

Ans. Official Answer NTA (25)

Sol. Let $f(x) = (x - \alpha)(x - \beta)$

It is given that $f(0) = p \Rightarrow \alpha\beta = p$

and $f(1) = \frac{1}{3} \Rightarrow (1-\alpha)(1-\beta) = \frac{1}{3}$

Now, let us assume that α is the common root of f(x) = 0 and fofofof(x) = 0

fofofof(x) = 0

⇒ fofof(0) = 0

⇒ fof(p) = 0

So, f(p) is either α or β.

$(p - \alpha)(p - \beta) = \alpha$

$(\alpha\beta - \alpha)(\alpha\beta - \beta) = \alpha \Rightarrow (\beta - 1)(\alpha - 1)\beta = 1$

$(\because \alpha \neq 0)$

So, β = 3

$(1-\alpha)(1-3)=\frac{1}{3}$

$\alpha=\frac{7}{6}$

$f(x)=\left(x-\frac{7}{6}\right)(x-3)$

$f(-3)=\left(-3-\frac{7}{6}\right)(3-3)=25$

Question ID : 1569423

Matrices

23. Let $A=\begin{bmatrix}1 & a & a\\0 & 1 & b\\0 & 0 & 1\end{bmatrix}$, a, b ∈ R. If for some n ∈ N, $A^n=\begin{bmatrix}1 & 48 & 2160\\0 & 1 & 96\\0 & 0 & 1\end{bmatrix}$ then n + a + b is equal to ________.

माना $A=\begin{bmatrix}1 & a & a\\0 & 1 & b\\0 & 0 & 1\end{bmatrix}$, a, b ∈ R है। यदि किसी n ∈ N के लिए $A^n=\begin{bmatrix}1 & 48 & 2160\\0 & 1 & 96\\0 & 0 & 1\end{bmatrix}$ है, तो n + a + b बराबर है ________

Ans. Official Answer NTA (24)

Sol. $A=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}+\begin{bmatrix}0 & a & a\\0 & 0 & b\\0 & 0 & 0\end{bmatrix}=I+B$

$A=\begin{bmatrix}0 & a & a\\0 & 0 & b\\0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & a & a\\0 & 0 & b\\0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & ab\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

$B^3=0$

$\therefore A^n=(1+B)^n={}^nC_0 I+{}^nC_1 B+{}^nC_2 B^2+{}^nC_3 B^3+\ldots\ldots$

$=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}+\begin{bmatrix}0 & na & na\\0 & 0 & nb\\0 & 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 0 & \frac{n(n-1)ab}{2}\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$

$$\begin{bmatrix} 1 & na & na+\frac{n(n-1)ab}{2} \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 48 & 2160 \\ 0 & 1 & 48 \\ 0 & 0 & 1 \end{bmatrix}$$

On compaing we get na = 48, nb = 96 and

$$na + \frac{n(n+1)}{2}ab = 2160$$

$\Rightarrow$ a= 4, n = 12 and b = 8

n + a + b = 24

Question ID : 1569424

Maxima & Minima

24. The sum of the maximum and minimum values of the function $f(x) = |5x - 7| + [x^2 + 2x]$ in the interval $\left[\frac{5}{4}, 2\right]$, where [t] is the greatest integer $\leq$ t, is _________.

फलन $f(x) = |5x - 7| + [x^2 + 2x]$, जहाँ [t] महत्तम पूर्णांक $\leq$ t है, के अन्तराल $\left[\frac{5}{4}, 2\right]$ में उच्चतम तथा निम्नतम मानों का योगफल है _________.

Ans. Official Answer NTA (15)

Sol. $f(x) = [5x - 7] + [x^2 + 2x]$

$= [5x - 7] + [(x + 1)^2] - 1$

Critical points of

$$f(x) = \frac{7}{5}\sqrt{5}-1, \sqrt{6}-1, \sqrt{7}-1, \sqrt{8}-1, 2$$

$\therefore$ Maximum or minimum value of f(X) occur at critical points or boundary points

$$\therefore \quad f\left(\frac{5}{4}\right) = \frac{3}{4} + 4 = \frac{19}{4}$$

$$f\left(\frac{7}{5}\right) = 0 + 4 = 4$$

as both $|5x - 7|$ and $x^2 + 2x$ are increasing in natuyre after $x = \frac{7}{5}$

$\therefore \quad f(2) = 3 + 8 = 11$

$\therefore \quad f\left(\frac{7}{5}\right)_{min} = 4$ and $f(2)_{max} = 11$

Sum is 4 + 11 = 15

Question ID : 1569425

Differential Equation

25. Let y = y(x) be the solution of the differential equation $\frac{dy}{dx} = \frac{4y^3 + 2yx^2}{3xy^2 + x^3}, y(1) = 1$. If for some $n \in N$, $y(2) \in [n-1, n)$, then n is equal to _________.

माना अवकल समीकरण $\frac{dy}{dx} = \frac{4y^3 + 2yx^2}{3xy^2 + x^3}, y(1) = 1$ का हल y = y(x) है। यदि किसी $n \in N$ के लिए $y(2) \in [n-1, n)$ है, तो n बराबर है _________

Ans. Official Answer NTA (3)

Sol. $\frac{dy}{dx} = \frac{y}{x}\frac{\left(4y^2 + 2x^2\right)}{\left(3y^2 + x^2\right)}$

Put y = vx $\quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = \frac{v\left(4v^2 + 2\right)}{\left(3v^2 + 1\right)}$

$\Rightarrow x\frac{dv}{dx} = v\left(\frac{\left(4v^2 + 2 - 3v^2 - 1\right)}{3v^2 + 1}\right)$

$\int\left(3v^2 + 1\right)\frac{dv}{v^3 + v} = \int\frac{dx}{x}$

$\Rightarrow \ln|v^3 + v| = \ln x + c$

$\Rightarrow \left|\left(\frac{y}{x}\right)^3 + \left(\frac{y}{x}\right)\right| = \ln x + C$

$\downarrow y(1) = 1$

$\Rightarrow C = \ln 2$

$\therefore$ for y(2)

$\ln\left(\frac{y^3}{8}+\frac{y}{2}\right)=2\ln 2 \Rightarrow \frac{y^3}{8}+\frac{y}{2}=4$

$\Rightarrow [y(2)] = 2$

$\Rightarrow n = 3$

Question ID : 1569426

Definite Integration

26. Let f be a twice differentiable function on R. If $f'(0) = 4$ and

$f(x)+\int_0^x (x-t)f'(t)dt = (e^{2x}+e^{-2x})\cos 2x + \frac{2}{a}x$, then $(2a + 1)^5 a^2$ is equal to ________.

माना R पर दो बार अवकलनीय एक फलन f है। यदि $f'(0) = 4$ है तथा

$f(x)+\int_0^x (x-t)f'(t)dt = (e^{2x}+e^{-2x})\cos 2x + \frac{2}{a}x$ है, तो $(2a + 1)^5 a^2$ बराबर है ________

Ans. Official Answer NTA (8)

Sol. $\because\ f(x)+\int_0^x (x-t)f'(t)dt = (e^{2x}+e^{-2x})$

$\cos 2x + \frac{2x}{a}$ .....(i)

Here $f(0) = 2$ .....(ii)

On differentiating equation (i) w.r.t. x we get :

$f'(x)+\int_0^x f'(t)dt + xf'(x) - xf'(x) = 2(e^{2x}-e^{-2x})$

$\cos 2x - 2(e^{2x}+e^{-2x})\sin 2x + \frac{2}{a}$

$\Rightarrow f'(x)+f(x)-f(0) = 2(e^{2x}-e^{-2x})\cos 2x - 2(e^{2x}+e^{-2x})$

$\sin 2x + \frac{2}{a}$

Replace x by 0 we get

$\Rightarrow\ 4 = \frac{2}{a} \quad \Rightarrow a = \frac{1}{2}.$

$\therefore\ (2a+1)^5 . a^2 = 2^5 . \frac{1}{2^2} = 2^3 = 8$

Question ID : 1569427

Definite Integration

27. Let $a_n = \int_{-1}^{n}\left(1+\frac{x}{2}+\frac{x^2}{3}+.....+\frac{x^{n-1}}{n}\right)dx$ for every $n \in N$. Then the sum of all the elements of the set $\{n \in N : a_n \in (2, 30)\}$ is _________.

माना प्रत्येक $n \in N$ के लिए $a_n = \int_{-1}^{n}\left(1+\frac{x}{2}+\frac{x^2}{3}+.....+\frac{x^{n-1}}{n}\right)dx$ है। तो समुच्चय $\{n \in N : a_n \in (2, 30)\}$ के सभी अवयवों का योगफल है _________

Ans. Official Answer NTA (5)

Sol. $\because a_n = \int_{-1}^{n}\left(1+\frac{x}{2}+\frac{x^2}{3}+.....+\frac{x^{n-1}}{n}\right)dx$

$= \left[x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+......+\frac{x^n}{n^2}\right]_{-1}^{n}$

$a_n = \frac{n+1}{1^2}+\frac{n^2-1}{2^2}+\frac{n^3+1}{3^2}+\frac{n^4-1}{4^2}+.....+\frac{n^n+(-1)^{n+1}}{n^2}$

Here $a_1 = 2,\ a_2 = \frac{2+1}{1}+\frac{2^2-1}{2} = 3+\frac{3}{2} = \frac{9}{2}$

$a_3 = 4+2+\frac{28}{9} = \frac{100}{9}$

$a_4 = 5+\frac{15}{4}+\frac{65}{9}+\frac{255}{16} > 31$

$\therefore$ The required set is $\{2, 3\}$. $\because a_n \in (2, 30)$

$\therefore$ Sum of elements = 5.

Question ID : 1569428

Circle

28. If the circles $x^2 + y^2 + 6x + 8y + 16 = 0$ and $x^2 + y^2 + 2\left(3-\sqrt{3}\right)x + 2\left(4-\sqrt{6}\right)y = k + 6\sqrt{3} + 8\sqrt{6}$, $k > 0$, touch internally at the point $P(\alpha, \beta)$, then $\left(\alpha+\sqrt{3}\right)^2 + \left(\beta+\sqrt{6}\right)^2$ is equal to _________.

यदि वृत्त $x^2 + y^2 + 6x + 8y + 16 = 0$ तथा $x^2+y^2+2\left(3-\sqrt{3}\right)x+2\left(4-\sqrt{6}\right)y=k+6\sqrt{3}+8\sqrt{6}$, $k>0$ बिंदु $P(\alpha, \beta)$ पर अंतः स्पर्श करते हैं, तो $\left(\alpha+\sqrt{3}\right)^2+\left(\beta+\sqrt{6}\right)^2$ बराबर है _________

Ans. Official Answer NTA (25)

Sol. The circle $x^2 + y^2 + 6x + 8y + 16 = 0$ has centre (–3, –4) and radius 3 units.

The circle $x^2+y^2+2(3-\sqrt{3})x+2(4-\sqrt{6})y=k+6\sqrt{3}+8\sqrt{6}, k>0$ has centre $\left(\sqrt{3}-3,\sqrt{6}-4\right)$ and radius $\sqrt{k+34}$

$\because$ These two circles touch internally hence

$\sqrt{3+6}=\left|\sqrt{k+34}-3\right|$

Here, $k = 2$ is only possible ($\because$ $k > 0$)

Equation of common tangent to two circles is $2\sqrt{3}x+2\sqrt{6}y+16+6\sqrt{3}+8\sqrt{6}+k=0$

$\because$ $k = 2$ then equation is

$x+\sqrt{2}y+3+4\sqrt{2}+3\sqrt{3}=0$ ..... (i)

$\because$ (α, β) are foot of perpendicular from (–3, –4)

To line (i) then

$$\frac{\alpha+1}{1}=\frac{\beta+4}{\sqrt{2}}=\frac{-\left(-3-4\sqrt{2}+3+4\sqrt{2}+3\sqrt{3}\right)}{1+2}$$

$$\therefore \alpha+3=\frac{\beta+4}{\sqrt{2}}=-\sqrt{3}$$

$$\Rightarrow \left(\alpha+\sqrt{3}\right)^2=9 \text{ and } \left(\beta+\sqrt{6}\right)^2=16$$

$$\therefore \left(\alpha+\sqrt{3}\right)^2+\left(\beta+\sqrt{6}\right)^2=25$$

Question ID : 1569429

Tangent and normal

29. Let the area enclosed by the x-axis, and the tangent and normal drawn to the curve $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$ at the point (–2, 3) be A. Then 8A is equal to _________.

माना वक्र $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$ के बिन्दु (–2, 3) पर खींची गई स्पर्श रेखा तथा अभिलंब और x-अक्ष से घिरे क्षेत्र का क्षेत्रफल A है। तो 8A बराबर है _________

Ans. Official Answer NTA (170)

Sol. $4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$

differentiating both sides we get

$12x^2 - 3y^2 - 6xyy' + 12x - 5y - 5xy' - 16yy' + 9 = 0$

$\downarrow (-2, 3)$

$\Rightarrow 48 - 27 + 36y' - 24 - 15 + 10y' - 48y' + 9 = 0$

$\Rightarrow 29' = - y$

$\Rightarrow m_T = \frac{-9}{2}$ & $m_N = \frac{2}{9}$

$T \equiv y - 3 = \frac{-9}{2}(x+2)$ & $N \equiv y - 3 = \frac{2}{9}(x+2)$

$\downarrow y = 0$ $\qquad\qquad \downarrow y = 0$

$x = \frac{-4}{3}$ $\qquad\qquad x = \frac{-31}{2}$

$\therefore$ Area $= \frac{1}{2} \times \text{Base} \times \text{Height}$

$= 8A = 170$

Question ID : 1569430

ITF

30. Let $x = \sin(2\tan^{-1}\alpha)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$. If $S = \{\alpha \in R : y^2 = 1 - x\}$, then $\sum_{\alpha \in S} 16\alpha^3$ is equal to __________.

माना $x = \sin(2\tan^{-1}\alpha)$ तथा $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$ हैं। यदि $S = \{\alpha \in R : y^2 = 1 - x\}$ है, तो $\sum_{\alpha \in S} 16\alpha^3$ बराबर है

____

Ans. Official Answer NTA (130)

Sol. $\because x = \sin\left(2\tan^{-1}\alpha\right) = \frac{2\alpha}{1+\alpha^2}$ ...(i)

and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right) = \sin\left(\sin^{-1}\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}$

Now, y2 = 1 – x

$\frac{1}{5} = 1 - \frac{2\alpha}{1+\alpha^2}$

$\Rightarrow 1 + \alpha^2 = 5 + 5\alpha^2 - 10\alpha$

$\Rightarrow 2\alpha^2 - 5\alpha + 2 = 0$

$\therefore \alpha = 2, \frac{1}{2}$

$\therefore \sum_{\alpha \in S} 16\alpha^3 = 16 \times 2^3 + 16 \times \frac{1}{2^3}$

$= 130$