JEE Main June 2022

Question Paper With Text Solution

24 June | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JUNE 2022 | 24^{TH} JUNE SHIFT-2

SECTION – A

Question ID : 131

Inverse Trigonometric Function

1. Let $x*y = x^2 + y^3$ and $(x*1)*1 = x*(1*1)$. Then a value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is :

माना $x*y = x^2 + y^3$ तथा $(x*1)*1 = x*(1*1)$ हैं। तो $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ का एक मान है :

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{6}$

Ans. Official Answer NTA (2)

Sol. given $x * y = x^2 + y^3$

Now

$(x * 1) * 1 = x * (1 * 1)$

Using the given condition

$(x^2 + 1^3) * 1 = x * (1^2 + 1^3)$

$(x^2 + 1) * 1 = x * (2)$

$((x^2 + 1)^2 + 1) = x^2 + 8$

$x^4 + 2x^2 + 2 = x^2 + 8$

$x^4 + x^2 - 6 = 0$

$(x^2 + 3)(x^2 - 2) = 0$

$x^2 = 2$ is possible answer

Now

$$2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$$

$$2\sin^{-1}\left(\frac{4}{8}\right) = \frac{\pi}{3}$$

Question ID : 132

Quadratic Equation

2. The sum of all the real roots of the equation $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ is :

समीकरण $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ के सभी वास्तविक मूलों का योग है :

(1) $\log_e 3$ (2) $-\log_e 3$ (3) $\log_e 6$ (4) $-\log_e 6$

Ans. Official Answer NTA (2)

Sol. $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$

$e^{2x} - 4 = 0$ $6e^{2x} - 5e^x + 1 = 0$

$e^{2x} = 4$ $e^x = \frac{1}{2}$

$e^{2x} = 4$ $e^x = \frac{1}{3}$

Now if we consider
roots as x_1, x_2, x_3 then

$e^{x_1} = \frac{1}{2}$

$e^{x_2} = \frac{1}{3}$

$e^{2x_3} = 4$

take logrithm on base 'e' and add

$x_1 + x_2 + x_3 = \ell n\left(\frac{1}{2}\right) + \ell n\left(\frac{1}{3}\right) + \frac{1}{2}\ell n(4)$

$x_1 + x_2 + x_3 = \ell n\left(\frac{1}{2}.\frac{1}{3}.2\right)$

$x_1 + x_2 + x_3 = \ell n\left(\frac{1}{3}\right)$

$= -ln3$

Question ID : 133

Straight Line

3. Let the system of linear equations

$x + y + \alpha z = 2$

$3x + y + z = 4$

$x + 2z = 1$

have a unique solution $(x*, y*, z*)$. If $(\alpha, x*)$, $(y*, \alpha)$ and $(x*, -y*)$ are collinear points, then the sum of absolute values of all possible values of α is :

माना रैखिक समीकरण निकाय

$x + y + \alpha z = 2$

$3x + y + z = 4$

$x + 2z = 1$

का केवल एक हल $(x*, y*, z*)$ है। यदि $(\alpha, x*)$, $(y*, \alpha)$ तथा $(x*, -y*)$ संरेख हैं, है तो α के सभी संभव मानों के निरपेक्ष मानों का योगफल है :

(1) 4 (2) 3 (3) 2 (4) 1

Ans. Official Answer NTA (3)

Sol. $\Delta = -(\alpha + 3)$

$D_x = -(3 + \alpha)$

$D_y = -(\alpha + 3)$

$D_z = 0$

for unique solutions $\alpha \neq -3$

Now

$x* = 1$

$y* = 1$

$z* = 0$

Now points are collinear

$$\begin{vmatrix} 1 & \alpha & 1 \\ 1 & 1 & \alpha \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$1(-1-\alpha) - \alpha(-1-\alpha) + 1(1-1) = 0$

$-1 + \alpha^2 = 0$

$\alpha^2 = 1$

$\alpha = \pm 1$

sum of absolute values = 1 + 1 = 2

Question ID : 134

Sequence and progression

4. Let x, y > 0. If $x^3 y^2 = 2^{15}$, then the least values of 3x + 2y is :

माना x, y > 0 हैं। यदि $x^3 y^2 = 2^{15}$ है, तो 3x + 2y का न्यूनतम मान है :

(1) 30 (2) 32 (3) 36 (4) 40

Ans. Official Answer NTA (4)

Sol. $x^3y^2 = 2^{15}$

Let consider numbers x, x, x, y, y and apply AM & GM

$$\frac{x+x+x+y+y}{5} \geq \left(x^3.y^2\right)^{\frac{1}{5}}$$

$$3x+2y \geq 5\left(2^{15}\right)^{\frac{1}{5}}$$

$3x + 2y \geq 40$

Then least value of 3x + 2y is 40.

Question ID : 135

Continuity & Differentiability

5. Let $f(x) = \begin{cases} \dfrac{\sin(x-[x])}{x-[x]} & , \quad x \in (-2,-1) \\ \max\{2x, 3[|x|]\} & , \quad |x| < 1 \\ 1 & , \quad \text{otherwise} \end{cases}$

Where [t] denotes greatest integer ≤ t. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is :

माना $f(x)=\begin{cases}\dfrac{\sin(x-[x])}{x-[x]} & , \quad x\in(-2,-1)\\ \max\{2x,3[|x|]\} & , \quad |x|<1\\ 1 & , \quad \text{otherwise}\end{cases}$

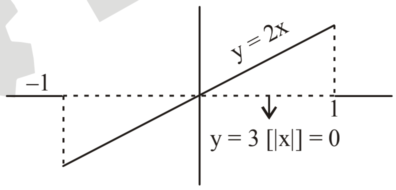
जहाँ [t] महत्तम पूर्णांक $\le t$ है। यदि m बिन्दुओं की संख्या है, जहाँ f संतत नहीं है, तथा n उन बिन्दुओं की संख्या है, जहाँ f अवकलनीय नहीं है, तो क्रमित युग्म (m, n) है :

(1) (3, 3) (2) (2, 4) (3) (2, 3) (4) (3, 4)

Ans. Official Answer NTA (3)

Sol. $f(x)=\begin{cases}\dfrac{\sin\{x\}}{\{x\}} & : \quad x\in(-2,-1)\\ \max(2x,3[|x|]) & : \quad -1<x<1\\ 1 & : \quad \text{otherwise}\end{cases}$

Let consider $y = \max(2x_1, 3[|x|]) : -1 , x < 1$



$y = \max(2x, 0) = \begin{cases}0 & : \quad -1<x<0\\ 2x & : \quad 0\le x<1\end{cases}$

Now doubtful points for continuity and differentiability are x = 0, 1, –1

Now continuity at x = 0

RHL = 0

f(0) = 0

LHL = 0

Now RHD = 2

LHD = 0

Continuity at x = 1

LHL = 2

RHL = 1

f(1) = 1

Continuity at x = –1

RHL = 0

$$\text{LHL} = \frac{\sin(1-h)}{1-h} = \sin 1$$

f(–1) = 1

hence it is clear function is not continuous at x = 1, – 1 and not differentiable at x = 1, –1, 0
m = 2
n = 3

(2, 3)

Question ID : 136

Definite Integration

6. The value of the integral $\int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)}$ is equal to :

समाकलन $\int_{-\pi/2}^{\pi/2} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)}$ का मान बराबर है :

(1) 2π (2) 0 (3) π (4) $\frac{\pi}{2}$

Ans. Official Answer NTA (3)

Sol. $$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(1+e^x)(\sin^6 x + \cos^6 x)} \quad \text{———(1)}$$

using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{(1+e^{-x})(\sin^6 x + \cos^6 x)}$$

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^x dx}{(e^x+1)(\sin^6 x + \cos^6 x)} \quad \text{———(2)}$$

by (1) + (2)

$$2I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dx}{\sin^6 x+\cos^6 x}$$

$$2I=2\int_{0}^{\frac{\pi}{2}}\frac{dx}{\sin^6 x+\cos^6 x}$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{dx}{1-3\sin^2 x\cos^2 x}$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{dx}{1-\frac{3}{4}\sin^2 2x}$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{4dx}{4-3\sin^2 2x}$$

$a\int_0^{2a} f(x)\,dx=2\int_0^a f(dx)$ If $f(2a-x)=f(x)$

$$I=2\int_0^{\pi/4}\frac{4.dx}{4-3\sin^2 2x}$$

$$I=2\int_0^{\pi/4}\frac{4\sec^2 2x.dx}{4\sec^2 2x-3\tan^2 2x}$$

$$I=2\int_0^{\pi/4}\frac{4\sec^2 2x.dx}{4+\tan^2 2x}$$

Put $\tan 2x=t$;

$2\sec^2 2x.dx=dt$

$$I=4\int_0^{\infty}\frac{dt}{4+t^2}$$

$$I=4\times\frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^{\infty}$$

$$I=2\left[\tan^{-1}\infty-\tan^{-1}0\right]$$

$$I=2\left[\frac{\pi}{2}-0\right]$$

$=\pi$

Question ID : 137

Definite Integration

7. $\lim_{n\to\infty}\left(\frac{n^2}{(n^2+1)(n+1)}+\frac{n^2}{(n^2+4)(n+2)}+....+\frac{n^2}{(n^2+n^2)(n+n)}\right)$ is equal to :

$\lim_{n\to\infty}\left(\frac{n^2}{(n^2+1)(n+1)}+\frac{n^2}{(n^2+4)(n+2)}+....+\frac{n^2}{(n^2+n^2)(n+n)}\right)$ बराबर है :

(1) $\frac{\pi}{8}+\frac{1}{4}\log_e 2$ (2) $\frac{\pi}{4}+\frac{1}{8}\log_e 2$ (3) $\frac{\pi}{4}-\frac{1}{8}\log_e 2$ (4) $\frac{\pi}{8}+\log_e\sqrt{2}$

Ans. Official Answer NTA (1)

Sol. $Tr=\frac{n^2}{(n^2+r^2)(n+r)}$

$$Tr=\frac{\frac{1}{n}}{\left(1+\frac{r^2}{n^2}\right)\left(1+\frac{r}{n}\right)}$$

$$Sum=\sum_{r=1}^{n}\frac{\frac{1}{n}}{\left(1+\frac{r^2}{n^2}\right)\left(1+\frac{r}{n}\right)}$$

$$=\int_0^1\frac{dx}{(1+x^2)(1+x)}$$

$$\frac{1}{(1+x^2)(1+x)}=\frac{A}{1+x}+\frac{Bx+C}{1+x^2}$$

$1 = A(1 + x^2) + (Bx + C)(1 + x)$

Compare

$$A=\frac{1}{2},\ B=-\frac{1}{2},\ C=\frac{1}{2}$$

$$=\int_0^1\left(\frac{\frac{1}{2}}{1+x^2}dx\right)+\int_0^1\frac{\left(-\frac{1}{2}x+\frac{1}{2}\right)}{\left(1+x^2\right)}dx$$

$$=\frac{1}{2}\int_0^1\frac{dx}{1+x}-\frac{1}{4}\int_0^1\frac{2xdx}{1+x^2}+\frac{1}{2}\int_0^1\frac{dx}{1+x^2}$$

$$=\frac{1}{2}\ell n\left(1+x\right)\Big]_0^1-\frac{1}{4}\ell n\left(1+x^2\right)\Big]_0^1+\frac{1}{2}\left[\tan^{-1}x\right]_0^1$$

$$=\frac{1}{2}\ell n2-\frac{1}{4}\ell n2+\frac{\pi}{8}$$

$$=\frac{\pi}{8}+\frac{1}{4}\ell n2$$

Question ID : 138

Differential Equation

8. A particle is moving in the xy-plane along a curve C passing through the point (3, 3). The tangent to the curve C at the point P meets the x-axis at Q. If the y-axis bisects the segment PQ, then C is a parabola with :

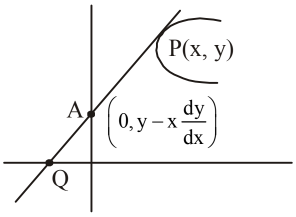
(1) length fo latus rectum 3
(2) length of latus rectum 6
(3) focus $\left(\frac{4}{3},0\right)$
(4) focus $\left(0,\frac{3}{4}\right)$

xy-समतल में कण, बिन्दु (3, 3) से होकर जाने वाले एक वक्र C के अनुदिश चल रहा है। माना वक्र C के बिन्दु P पर स्पर्श रेखा, x-अक्ष को बिन्दु Q पर मिलती है। यदि रेखाखण्ड PQ को y-अक्ष समद्विभाजित करता है, तो C एक परवलय है जिसकी:

(1) नाभिलंब जीवा की लम्बाई 3 है
(2) नाभिलंब जीवा की लम्बाई 6 है
(3) नाभी $\left(\frac{4}{3},0\right)$ है
(4) नाभी $\left(0,\frac{3}{4}\right)$ है

Ans. Official Answer NTA (1)

Sol.



tangent at 'P'

$$Y - y = \frac{dy}{dx}(X - x)$$

$A\left(-y\frac{dx}{dy} + x, 0\right), Q\left(x - \frac{y}{dy/dx}, 0\right), P(x, y)$

Since A is mid point of segment PQ

$$\frac{x - y\frac{dx}{dy} + x}{1+1} = 0$$

$$y\frac{dx}{dy} = 2x$$

$$\int 2\frac{dy}{y} = \int \frac{dx}{x}$$

$2\,ln y = ln x + ln c$

$2 ln y = ln cx$

$y^2 = cx$

Curve passing through (3, 3) then

$9 = 3c$

$c = 3$

$y^2 = 3x$

Length of latus rectum is 3.

Question ID : 139

Maxima and Minima

9. Let the maximum area of the triangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$, $a > 2$, having one of its vertices at one end of the major axis of the ellipse and one of its sides parallel to the y-axis, be $6\sqrt{3}$. Then the eccentricity of the ellipse is :

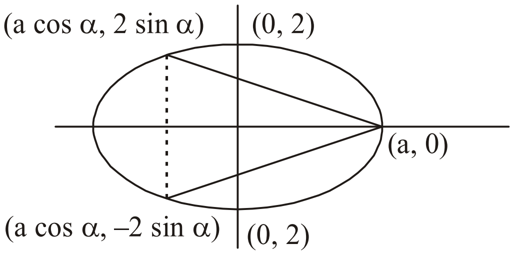
माना दीर्घवृत्त $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$, $a > 2$ के अन्तर्गत त्रिभुज, जिसका एक शीर्ष दीर्घवृत्त के दीर्घअक्ष के एक सिरे पर है तथा

जिसकी एक भुजा y-अक्ष के समांतर है, का अधिकतम क्षेत्रफल $6\sqrt{3}$ है। तो दीर्घवृत्त की उत्केन्द्रता है :

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{2}$ (3) $\frac{1}{\sqrt{2}}$ (4) $\frac{\sqrt{3}}{4}$

Ans. Official Answer NTA (1)

Sol.



Are of triangle = $\frac{1}{2}\left|4\sin\alpha\left(a-a\cos\alpha\right)\right|$

$=\frac{1}{2}\left|4a\sin\alpha\left(1-\cos\alpha\right)\right|$

Let consider f(α) = sin α (1 – cos α)

Minimum value of f(α) will be at $\alpha=\frac{2\pi}{3}$

$$f\left(\alpha\right)_{max}=\frac{3\sqrt{3}}{4}$$

maximum area of $\Delta=\frac{1}{2}\left|4a\frac{3\sqrt{3}}{4}\right|=6\sqrt{3}$

$e=\sqrt{1-\frac{b^2}{a^2}}$ a = 4

$e=\sqrt{1-\frac{4}{16}}$

$=\sqrt{1-\frac{1}{4}}=\frac{\sqrt{3}}{2}$

Question ID : 1310

Straight Line

10. Let the area of the triangle with vertices A(1, α), B(α, 0) and C(0, α) be 4 sq. units. If the points (α, –α), (–α, α) and (α^2, β) are collinear, then β is equal to :

माना शीर्ष A(1, α), B(α, 0) तथा C(0, α) के त्रिभुज का क्षेत्रफल 4 वर्ग इकाई है। यदि बिन्दु (α, –α), (–α, α) तथा (α^2, β) संरेख हैं, तो β बराबर है :

(1) 64 (2) –8 (3) –64 (4) 512

Ans. Official Answer NTA (3)

Sol. Area of triangle $\left|\frac{1}{2}\begin{vmatrix} 1 & 1 & \alpha \\ 1 & \alpha & 0 \\ 1 & 0 & \alpha \end{vmatrix}\right| = 4$

$\alpha = \pm 8$

Now given points are collinear then

$$\begin{vmatrix} \alpha & -\alpha & 1 \\ -\alpha & \alpha & 1 \\ \alpha^2 & \beta & 1 \end{vmatrix} = 0$$

If put $\alpha = +8$ or $\alpha = -8$ in above determinant we get

$\beta = -64$

Question ID : 1311

Monotoncity

11. The number of distinct real roots of the equation $x^7 - 7x - 2 = 0$ is :

समीकरण $x^7 - 7x - 2 = 0$ के भिन्न वास्तविक मूलों की संख्या है :

(1) 5 (2) 7 (3) 1 (4) 3

Ans. Official Answer NTA (4)

Sol. $x^7 - 7x - 2 = 0$

Consider

$f(x) = x^7 - 7x - 2$

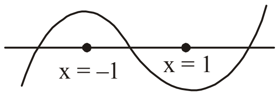
$f'(x) = 7x^6 - 7$

$f'(x) = 7(x^6 - 1)$

$= 7(x^2 - 1)(x^4 + x^2 + 1)$

$= 7(x+1)(x-1)(x^4 + x^2 + 1)$

If we scatch graph of f(x)



Clearly number of solutions are 3

Question ID : 1312

Probability

12. A random variable X has the following probability distribution :

x	0	1	2	3	4
P(X)	k	2k	4k	6k	8k

The value of $P(1 < X < 4 \mid X \le 2)$ is equal to :

एक यादृच्छिक चर X का प्रायिकता बंटन निम्न है :

x	0	1	2	3	4
P(X)	k	2k	4k	6k	8k

$P(1 < X < 4 \mid X \le 2)$ का मान बराबर है :

(1) $\frac{4}{7}$ (2) $\frac{2}{3}$ (3) $\frac{3}{7}$ (4) $\frac{4}{5}$

Ans. Official Answer NTA (1)

Sol. Let $1 < x < 4 = A$

$x \le 2 = B$

Now $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$= \frac{4k}{k + 2k + 4k}$

$= \frac{4k}{7k}$

$= \frac{4}{7}$

Question ID : 1313

Trigonometric Equation

13. The number of solutions of the equation $\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$, $x\in[-3\pi, 3\pi]$ is :

समीकरण $\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$, $x\in[-3\pi, 3\pi]$ के हलों की संख्या है :

(1) 8 (2) 5 (3) 6 (4) 7

Question ID : 1313

Ans. Official Answer NTA (4)

Sol. $\cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$

$\cos^2\frac{\pi}{3}-\sin^2 x=\frac{1}{4}\cos^2 2x$

$\frac{1}{4}-\sin^2 x=\frac{1}{4}(1-2\sin^2 x)^2$

$(1-4\sin^2 x)=(1-2\sin^2 x)^2$

$\sin^2 x = t$

$(1-4t)=(1-2t)^2$

$1-4t=4t^2-4t+1$

$t^2=0$

$\sin^2 x=0$

$x=-3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi$

hence total number of solution are 7

Question ID : 1314

3D Geometry

14. If the shortest distance between the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{\lambda}$ and $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-5}{5}$ is $\frac{1}{\sqrt{3}}$, then the sum of all possible values of λ is :

यदि रेखाओं $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{\lambda}$ तथा $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-5}{5}$ के बीच न्यूनतम दूरी $\frac{1}{\sqrt{3}}$ है, तो λ के सभी संभव मानों का योगफल है :

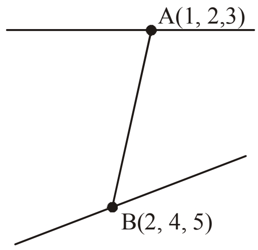
(1) 16 (2) 6 (3) 12 (4) 15

Ans. Official Answer NTA (1)

Sol. $\vec{p} = 2i + 3j + \lambda k$

$\vec{q} = i + 4j + 5k$

$\overrightarrow{AB} = i + 2j + 2k$



minimum distance

$$\left|\frac{\overrightarrow{AB}.(\vec{p}\times\vec{q})}{\vec{p}\times\vec{q}}\right| = \frac{1}{\sqrt{3}}$$

$$\frac{\left|\overrightarrow{AB}.(\vec{p}\times\vec{q})\right|}{\left|\vec{p}\times\vec{q}\right|} = \frac{1}{\sqrt{3}}$$

$$\overrightarrow{AB}.(\vec{p}\times\vec{q}) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & \lambda \\ 1 & 4 & 5 \end{vmatrix} = 5 - 2\lambda$$

$$\left|\vec{p}\times\vec{q}\right| = \left|i(15-4\lambda) - j(10-\lambda) + k(5)\right|$$

$= 17\lambda^2 - 140\lambda + 350$

Now

$$\frac{5-2\lambda}{\sqrt{17\lambda^2 - 140\lambda + 350}} = \frac{1}{\sqrt{3}}$$

$\lambda^2 - 16\lambda + 55 = 0$

Sum of roots = 16

Question ID : 1315

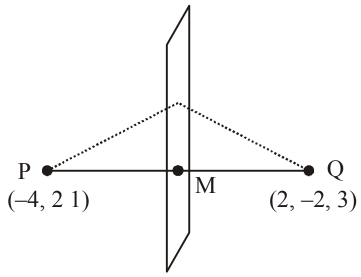
3D Geometry

15. Let the points on the plane P be equidistant from the points (–4, 2, 1) and (2, –2, 3). Then the acute angle between the plane P and the plane $2x + y + 3z = 1$ is :

माना समतल P के बिन्दु, बिन्दुओं (–4, 2, 1) तथा (2, –2, 3) के बराबर दूरी पर है। तो समतल P तथा समतल $2x + y + 3z = 1$ के बीच न्यूनकोण है :

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{4}$ (3) $\frac{\pi}{3}$ (4) $\frac{5\pi}{12}$

Ans. Official Answer NTA (3)

Sol. 

M (–1, 0, 2) is point on plane

Normal $\overrightarrow{PQ} = \vec{n} = 6i - 4j + 2k$

equation of required plane

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(6\hat{i} - 4\hat{j} + 2\hat{k}\right) = -6 + 4$

6x – 4y + 2z + 2 = 0

angle between two plane

$$\cos\theta = \left|\frac{12 - 4 + 6}{\sqrt{36+16+4} \times \sqrt{4+1+9}}\right|$$

$$= \left|\frac{14}{\sqrt{56} \times \sqrt{14}}\right|$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Question ID : 1316

Vectors

16. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that $\left|\left(\hat{a} + \hat{b}\right) + 2\left(\hat{a} \times \hat{b}\right)\right| = 2$. If $\theta \in (0, \pi)$ is the angle between $\hat{a}$ and $\hat{b}$, then among the statements :

(S1) : $2\left|\hat{a} \times \hat{b}\right| = \left|\hat{a} - \hat{b}\right|$

(S2) : The projection of $\hat{a}$ on $\left(\hat{a} + \hat{b}\right)$ is $\frac{1}{2}$

(1) only (S1) is true (2) only (S2) is true

(3) Both (S1) and (S2) are true (4) Both (S1) and (S2) are false

माना दो इकाई सदिशों $\hat{a}$ तथा $\hat{b}$ के लिए $\left|\left(\hat{a}+\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right)\right|=2$ है। यदि $\hat{a}$ तथा $\hat{b}$ के बीच का कोण $\theta \in (0, \pi)$ है, है, तो कथनों :

(S1) : $2\left|\hat{a}\times\hat{b}\right|=\left|\hat{a}-\hat{b}\right|$

(S2) : $\hat{a}$ का $\left(\hat{a}+\hat{b}\right)$ पर प्रक्षेप $\frac{1}{2}$ है

(1) केवल (S1) सत्य है (2) केवल (S2) सत्य है

(3) (S1) तथा (S2) दोनों सत्य हैं (4) (S1) तथा (S2) दोनों असत्य हैं

Ans. Official Answer NTA (3)

Sol. $\left|\left(\hat{a}+\hat{b}\right)+2\left(\hat{a}\times\hat{b}\right)\right|=2$

$\left(\hat{a}+\hat{b}\right)^2+4\left(\hat{a}+\hat{b}\right)^2+4\left(\vec{a}+\vec{b}\right)\left(\vec{a}\times\vec{b}\right)=4$

$1+1+2\cos\theta+4\sin^2\theta=4$

$1+\cos\theta+2\left(1-\cos^2\theta\right)=2$

$\cos\theta=\frac{-1}{2}$ is correct in $\theta \in (0, \pi)$

$\theta=\frac{2\pi}{3}$

(S1) $2\left|\hat{a}\times\hat{b}\right|=\left|\hat{a}-\hat{b}\right|$

$2\times\frac{\sqrt{3}}{2}=\sqrt{1+1+2\times\frac{1}{2}}$

$\sqrt{3}=\sqrt{3}$

which is correct

(S2) $\left|\frac{\hat{a}.\left(\hat{a}+\hat{b}\right)}{\left|\hat{a}+\hat{b}\right|}\right|=\frac{1}{2}$

$\left|\frac{1-\frac{1}{2}}{1}\right|=\frac{1}{2}$

which is also correct

Question ID : 1317

Methods of Differentiation

17. If $y=\tan^{-1}\left(\sec x^3-\tan x^3\right), \frac{\pi}{2}<x^3<\frac{3\pi}{2}$, then :

यदि $y=\tan^{-1}\left(\sec x^3-\tan x^3\right), \frac{\pi}{2}<x^3<\frac{3\pi}{2}$, है, तो :

(1) $xy''+2y'=0$

(2) $x^2y''-6y+\frac{3\pi}{2}=0$

(3) $x^2y''-6y+3\pi=0$

(4) $xy''-4y'=0$

Ans. Official Answer NTA (2)

Sol. $y=\tan^{-1}\left(\frac{1}{\cos x^3}-\frac{\sin x^3}{\cos x^3}\right)$

$y=\tan^{-1}\left(\frac{1-\sin x^3}{\cos x^3}\right)$

$y=\tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}-x^3\right)}{\sin\left(\frac{\pi}{2}-x^3\right)}\right)$

$y=\tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x^3}{2}\right)\right)$

$-\frac{\pi}{2}<\frac{\pi}{4}-\frac{x^3}{2}<0$

$y=\frac{\pi}{4}-\frac{x^3}{2}$

$\frac{dy}{dx}=\frac{-3x^2}{2}$

$y'=\frac{-3}{2}x^2$

$y''=-3x$

Consider option (A) $xy''+2y'=0$

$x(-3x)+2\left(-\frac{3}{2}x^2\right)=0$

which is wrong

(B) $x^2y''-6y+\frac{3\pi}{2}=0$

$x^2(-3x)-6\left(\frac{\pi}{4}-\frac{x^3}{2}\right)+\frac{3\pi}{2}=0$

which is correct option

Question ID : 1318

Mathematical Reasoning

18. Consider the following statements :

A : Rishi is a judge.

B : Rishi is honest.

C : Rishi is not arrogant.

The negation of the statement "if Rishi is a judge and he is not arrogant, then he is honest" is :

निम्न कथनों का विचार कीजिए :

A : रिषी एक न्यायाधीश है

B : रिषी ईमानदार है

C : रिषी घमंडी नहीं है

कथन "यदि रिषी एक न्यायाधीश है तथा वह घमंडी नहीं है, तो वह ईमानदार है" का निषेधन है :

(1) $B \to (A \vee C)$

(2) $(\sim B) \wedge (A \wedge C)$

(3) $B \to ((\sim A) \vee (\sim C))$

(4) $B \to (A \wedge C)$

Ans. Official Answer NTA (2)

Sol. given statement is

$(A \wedge C) \to B$

$\sim (A \wedge C) \vee B$

we know

$p \to q = \sim p \vee q$

negative of above

$(A \wedge C) \wedge \sim B$

$\sim B \wedge (A \wedge C)$

Question ID : 1319

Differential Equation

19. The slope of normal at any point (x, y), x > 0, y > 0 on the curve y = y(x) is given by $\frac{x^2}{xy - x^2y^2 - 1}$. If the curve passes through the point (1, 1), then e.y(e) is equal to :

वक्र y = y(x) के किसी भी बिन्दु (x, y), x > 0, y > 0 पर अभिलंब की प्रवणता $\frac{x^2}{xy - x^2y^2 - 1}$ द्वारा दी गई है। यदि यह वक्र बिन्दु (1, 1) से होकर जाता है, तो e.y(e) बराबर है :

(1) $\frac{1-\tan(1)}{1+\tan(1)}$ (2) tan(1) (3) 1 (4) $\frac{1+\tan(1)}{1-\tan(1)}$

Ans. Official Answer NTA (4)

Sol. $-\frac{dx}{dy} = \frac{x^2}{xy - x^2y^2 - 1}$

$\frac{dy}{dx} = -\left(\frac{xy - x^2y^2 - 1}{x^2}\right)$

$\frac{dy}{dx} = \left(\frac{x^2y^2 - xy + 1}{x^2}\right)$

$xy = v$

$y + x\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{x}\left(\frac{dv}{dx} - y\right)$

$\left(\frac{dv}{dx} - y\right) = \frac{v^2 - v + 1}{x}$

$\frac{dv}{dx} - \frac{v}{x} = \frac{v^2 - v + 1}{x}$

$x\frac{dv}{dx} = v^2 + 1$

$\int \frac{dv}{v^2 + 1} = \int \frac{dx}{x}$

$\tan^{-1}(v) = l\text{n}x + c$

$\tan^{-1}(xy) = l\text{n}x + c$

passing through (1, 1)

$\tan^{-1}1 = 0 + c$

$\tan^{-1}(xy) = l\text{n}x + \tan^{-1}(1)$

$xy = \tan(\ln x + \tan^{-1}(1))$

$$y = \frac{1}{x}\tan\left(\ell nx + \frac{\pi}{4}\right)$$

$$y(e) = \frac{1}{e}\tan\left(1 + \frac{\pi}{4}\right)$$

now

$$e.\, y(e) = \frac{1+\tan 1}{1-\tan 1}$$

Question ID : 1320

Monotonocity

20. Let λ^* be the largest value of λ for which the function $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$ is increasing for all $x \in R$. Then $f_{\lambda^*}(1) + f_{\lambda^*}(-1)$ is equal to :

माना λ^* का अधिकतम मान, जिसके लिए फलन $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$ सभी $x \in R$ के लिए वर्धमान है, λ^* है। तो $f_{\lambda^*}(1) + f_{\lambda^*}(-1)$ बराबर है :

(1) 36 (2) 48 (3) 64 (4) 72

Ans. Official Answer NTA (4)

Sol. $f_\lambda(x) = 4\lambda x^3 - 36dx^2 + 36x + 48$

$$f'_\lambda(x) = 12\lambda x^2 - 72\lambda x + 36$$

$$f'_\lambda(x) = 12(\lambda x^2 - 6\lambda x + 3)$$

since function is increasing

$\lambda > 0$ ——(1)

$\lambda < 0$

$36\lambda^2 - 12\lambda \le 0$ ——(2)

by (1) and (2)

$$\lambda \in \left(0, \frac{1}{3}\right]$$

$$f_{\frac{1}{3}}(x) = \frac{4}{3}x^3 - 12x^2 + 36x + 48$$

$$f_{\frac{1}{3}}(1)=\frac{4}{3}-12+36+48$$

$$f_{\frac{1}{3}}(-1)=\frac{-4}{3}-12-36+48$$

$$f_{\frac{1}{3}}(1)+f_{\frac{1}{3}}(-1)=72$$

SECTION – B

Question ID:1321

Complex number

21. Let $S=\{z\in C:|z-3|\le 1 \text{ and } z(4+3i)+\overline{z}(4-3i)\le 24\}$. If $\alpha + i\beta$ is the point in S which is closest to 4i, then 25 $(\alpha + \beta)$ is equal to ________.

माना $S=\{z\in C:|z-3|\le 1$ तथा $z(4+3i)+\overline{z}(4-3i)\le 24\}$ हैं। यदि S में 4i के निकटतम बिन्दु $\alpha + i\beta$ है, तो 25 $(\alpha + \beta)$ बराबर है ________

Ans. Official Answer NTA (80)

Sol. here $|z-3|\le 1$

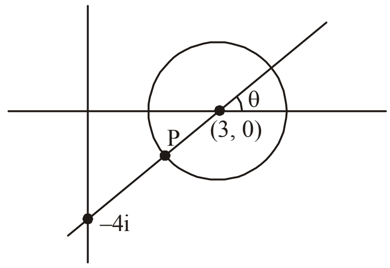
$(x-3)^2+y^2\le 1$

and

$$z(4+3i)+\overline{z}(4-3i)\le 24$$

$4x-3y\le 12$

$$\tan\theta=\frac{4}{3}$$



here we need point P

P$(3-\cos\theta, \sin\theta)$

$$P\left(3-\frac{3}{5},\frac{4}{5}\right)$$

$$P\left(\frac{12}{5},\frac{4}{5}\right)$$

$$\alpha+i\beta=\left(\frac{12}{5}+i\frac{4}{5}\right)$$

$$\alpha=\frac{12}{5}$$

$$\beta=\frac{4}{5}$$

$\alpha+\beta=\frac{16}{5}$

$25(\alpha+\beta)=80$

Question ID:1322

Matrices

22. Let $S=\left\{\begin{pmatrix}-1 & a\\ 0 & b\end{pmatrix}; a,b\in\{1,2,3,...100\}\right\}$ and let $T_n=\{A\in S: A^{n(n+1)}=I\}$. Then the number of elements in $\bigcap_{n=1}^{100} T_n$ is __________.

माना $S=\left\{\begin{pmatrix}-1 & a\\ 0 & b\end{pmatrix}; a,b\in\{1,2,3,...100\}\right\}$ तथा $T_n=\{A\in S: A^{n(n+1)}=I\}$ हैं। तो $\bigcap_{n=1}^{100} T_n$ में अवयवों की संख्या है __________.

Ans. Official Answer NTA (100)

Sol. $A=\begin{pmatrix}-1 & a\\ 0 & b\end{pmatrix}$

$A^2=\begin{pmatrix}1 & -a+ab\\ 0 & b^2\end{pmatrix}$

if b = 1

$A^2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

and $a\in\{1, 2, 3, ...100\}$

here n(n + 1) is always even

then $T_1, T_2, T_3 .....T_n$ are all I for

b = 1 and each value of a $\bigcap_{n=1}^{100} T_n=100$

Question ID:1323

P & C

23. The number of 7-digit numbers which are multiples of 11 and formed using all the digits 1, 2, 3, 4, 5, 7 and 9 is __________.

सभी अंको 1, 2, 3, 4, 5, 7 तथा 9 के प्रयोग से बनने वाली 7-अंकीय संख्याओं, जो 11 की गुणज हैं की संख्या है __________.

Ans. Official Answer NTA (576)

Sol. Divisible rule of 11

$$\left[\begin{pmatrix}\text{sum of digits}\\ \text{at odd place}\end{pmatrix}-\begin{pmatrix}\text{sum of digits at}\\ \text{even plance}\end{pmatrix}\right]$$ should be divisible by 11.

Given digits are 1, 2, 3, 4, 5, 7, 9 (sum = 31)

our number will be of type $a_1a_2a_3a_4a_5a_6a_7$

Now $(a_1 + a_3 + a_5 + a_7) - (a_2 + a_4 + a_6) = 0, 11, -11, 22, -22$ (not possible)

Case – I : $(a_1 + a_3 + a_5 + a_7) - (a_2 + a_4 + a_6) = 0$

$a_1 + a_3 + a_5 + a_7 = a_2 + a_4 + a_6$

which is not possible

Case – II : $(a_1 + a_3 + a_5 + a_7) - (a_2 + a_4 + a_6) = 11$

$(a_1 + a_3 + a_5 + a_7) = 11 + \underbrace{a_2 + a_4 + a_6}_{x}$

$31 - x = 11 + x$

$2x = 20$

$x = 10$

$a_2 + a_4 + a_6 = 10$

(i) (1, 4, 5) no. of ways = $\lfloor 3$

$a_1 + a_2 + a_5 + a_7 = 21$

(2, 3, 7, 9) no. of ways = $\lfloor 4$

Total = $\lfloor 4 \times \lfloor 3$

= 144

(ii) $a_2 + a_4 + a_6 = 10$

(1, 2, 7) no. of ways = $\lfloor 3$

$a_1 + a_3 + a_5 + a_7 = 21$

(3, 4, 5, 9) no. of ways = $\lfloor 4$

Total no. of ways = $\lfloor 4 \times \lfloor 3$

= 144

(iii) $a_2 + a_4 + a_6 = 10$

(2, 3, 5) no. of ways = $\lfloor 3$

$a_1 + a_3 + a_5 + a_7 = 21$

(1, 4, 7, 9) no. of ways = $\lfloor 4$

Total no. of ways = $\lfloor 4 \times \lfloor 3$

= 144

Case – II : $(a_1 + a_3 + a_5 + a_7) - (a_2 + a_4 + a_6) = -11$

$(a_1 + a_3 + a_5 + a_7) = -11 + \underbrace{a_2 + a_4 + a_6}_{x}$

$31 - x = -11 + x$

$2x = 42$

$x = 21$

$a_2 + a_4 + a_6 = 21$

(5, 7, 9) no. of ways = $\lfloor 3$

$a_1 + a_3 + a_5 + a_7 = 10$

(1, 2, 3, 4) no. of ways = $\lfloor 4$

Total $= \lfloor 4 \times \lfloor 3 = 144$

Final answer $= 144 + 144 + 144 + 144$

$= 576$

Question ID:1324

Sequence and progression

24. The sum of all the elements of the set $\{\alpha \in \{1, 2, ...100\} : \text{HCF}(\alpha, 24) = 1\}$ is _________.

समुच्चय $\{\alpha \in \{1, 2, ...100\} : \text{HCF}(\alpha, 24) = 1$ है$\}$ के सभी अवयवों का योगफल है _________.

Ans. Official Answer NTA (1633)

Sol. from set $\alpha \in \{1, 2, ...100\}$ numbers which are divisible by '2' or '3' cannot have HCF '1' with 24

number divisible by 2 = {2, 4, 6...... 100} = 50 numbers.

$$\text{sum} = \frac{50}{2}(2 + 100) = 50 \times 51$$

numbers divisible by 3 = {3, 6, ...99} = 33 numbers

$$\text{sum} = \frac{33}{2}(3 + 99) = 33 \times 51$$

numbers divisible by 6 = {6, 12, ...96} = 16 numbers

$$\text{sum} = \frac{16}{2}(6 + 96) = 16 \times 51$$

sum of all number = 1 + 2 + ... + 100 = 50 × 101

hence sum of number divisible by '2' or '3'

can be given as = 50 × 51 + 33 × 51 – 16 × 51

= 67 × 51

sum of required numbers = 50 × 101 – 67 × 51

$= 1633$

Question ID:1325

Binomial Theorem

25. The remainder on dividing $1 + 3 + 3^2 + 3^3 + ... + 3^{2021}$ by 50 is _________.

$1 + 3 + 3^2 + 3^3 + ... + 3^{2021}$ को 50 से विभाजित करने पर शेषफल है _________

Ans. Official Answer NTA (4)

Sol. $1+3+3^2+......+3^{2021} = \frac{1}{2}\left[3^{2022}-1\right]$

$=\frac{1}{2}\left[9^{1011}-1\right]$

$=\frac{1}{2}\left[(10-1)^{1011}-1\right]$

$=\frac{1}{2}\left[100k+10110-1-1\right]$

$= 50k + 5055 - 1$

$= 50k + 5054$

$= 50k + 5050 + 4$

$= 50k_1 + 4$

hence remainder is $=4$

Question ID:1326

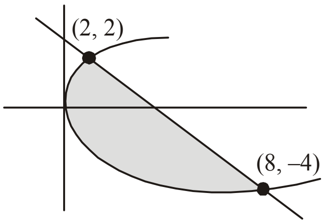
Area Under Curve

26. The area (in sq. units) of the region enclosed between the parabola $y^2 = 2x$ and the line $x + y = 4$ is _________.

परवलय $y^2 = 2x$ तथा रेखा $x + y = 4$ से घिरे क्षेत्र का क्षेत्रफल (वर्ग इकाईयों में) है _________

Ans. Official Answer NTA (18)

Sol.



Required area $= \int_{-4}^{2} (x \text{ line} - x \text{ parabola})\,dy$

$= \int_{-4}^{2} \left(4 - y - \frac{y^2}{2}\right) dy$

$= 4y - \frac{y^2}{2} - \frac{y^3}{6}\Big]_{-4}^{2}$

= 18 square units

Question ID:1327

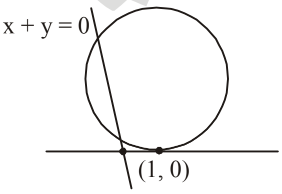
Circle

27. Let circle C : $(x - h)^2 + (y - k)^2 = r^2$, $k > 0$, touch the x-axis at (1,0). If the line $x + y = 0$ intersects the circle C at P and Q such that the length of the chord PQ is 2, then the value of $h + k + r$ is equal to __________.

माना एक वृत्त C : $(x - h)^2 + (y - k)^2 = r^2$, $k > 0$, x-अक्ष को (1,0) पर स्पर्श करता है। यदि रेखा $x + y = 0$ वृत्त C को बिन्दुओं P तथा Q पर काटती है तथा जीवा PQ की लम्बाई 2 है, तो $h + k + r$ का मान बराबर है __________

Ans. Official Answer NTA (7)

Sol.



By concept of family of circle equation of circle

$$(x - 1)^2 + (y - 0)^2 + \lambda(y - 0) = 0$$

intercept on this circle by

$x + y = 0$ is 2 then

$2\sqrt{r^2 - p^2} = 2$

$$2\sqrt{\frac{\lambda^2}{4}-\left(\frac{1-\frac{\lambda}{2}}{\sqrt{2}}\right)^2}=2$$

we get $\lambda = -6, 2$

$\lambda = -6$ according to given condation

Now circle

$$(x-1)^2 + y^2 - 6y = 0$$

$$(x-1)^2 + (y-3)^2 = 9$$

$h = 1$

$k = 3$

$r = 3$

$h + k + r = 7$

Question ID:1328

Probability

28. In an examination, there are 10 true-false type questions. Out of 10, a student can guess the answer of 4 questions correctly with probability 3/4 and the remaining 6 questions correctly with probability 1/4. If the probability that the student guesses the answeres of exactly 8 questions correctly out of 10 is $\frac{27k}{4^{10}}$, then k is equal to __________.

एक परीक्ष में सत्य-असत्य वाले 10 प्रश्न हैं। एक छात्र द्वारा 10 प्रश्नों में से 4 प्रश्नों के उत्तर के सही अनुमान की प्रायिकता 3/4 है तथा शेष 6 प्रश्नों के उत्तर के सही अनुमान की प्रायिकता 1/4 है। यदि छात्र द्वारा सभी 10 प्रश्नों के दिए गए उत्तरों में से ठीक 8 उत्तरों के अनुमान सही होने की प्रायिकता $\frac{27k}{4^{10}}$ है, तो k बराबर है __________

Ans. Official Answer NTA (479)

Sol. So we can say student solve only two wrong questions, so these are the possibilities

(i) Both wrong from first section

(ii) Both wrong from second section

(iii) One wrong from each section

Probabilities $= {}^4C_2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^6$

$+{}^6C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4$

$+{}^4C_1\times{}^6C_1\times\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^5$

$=\frac{27\times479}{4^{10}} = \frac{27\times k}{4^{10}} \Rightarrow k=479$

Question ID:1329

Hyperbola

29. Let the hyperbola H: $\frac{x^2}{a^2}-y^2=1$ and the ellipse E : $3x^2+4y^2+4y^2=12$ be such that the length of latus rectum of H is equal to the length of latus rectum of E. If e_H and e_E are the ecentricities of H and E respectively, then the value of $12\left(e_H^2+e_E^2\right)$ is equal to _________.

माना अतिपरवलय H : $\frac{x^2}{a^2}-y^2=1$ की नाभिलंब जीवा की लंबाई, दीर्घवृत्त E : $3x^2+4y^2+4y^2=12$ की नाभिलंब जीवा की लम्बाई के बराबर है। यदि H तथा E की उत्केन्द्रताएँ क्रमशः e_H तथा e_E है, तो $12\left(e_H^2+e_E^2\right)$ का मान बराबर है _________.

Ans. Official Answer NTA (42)

Sol. $\frac{x^2}{a^2}-\frac{y^2}{1}=1$ & $\frac{x^2}{4}+\frac{y^2}{3}=1$

By equations length of lates-reactum

$\frac{2\times1}{a}=\frac{2\times3}{2}$

$a=\frac{2}{3}$

$e_H=\sqrt{1+\frac{b^2}{a^2}}=\sqrt{1+\frac{1\times9}{9}}=\sqrt{\frac{13}{4}}$

$e_E=\sqrt{1-\frac{3}{4}}=\frac{1}{2}$

value $=12(e_H^2+e_E^2)$

$$= 12\left(\frac{13}{4} + \frac{1}{4}\right)$$

$$= 42$$

Question ID:1330

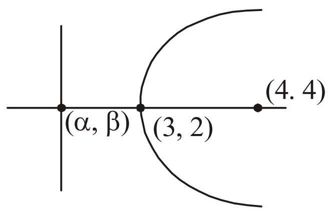
Parabola

30. Let P_1 be a parabola with vertex (3, 2) and focus (4, 4) and P_2 be its mirror image with respect to the line $x + 2y = 6$. Then the directrix of P_2 is $x + 2y =$ __________.

माना P_1 एक परवलय है जिसका शीर्ष (3, 2) है तथा नाभि (4, 4) है, तथा रेखा $x + 2y = 6$ के सापेक्ष P_1 का दर्पण प्रतिबिंब P_2 है। तो P_2 की नियता $x + 2y =$ __________ है।

Ans. Official Answer NTA (10)

Sol.



We know $\frac{\alpha+4}{2} = 3$

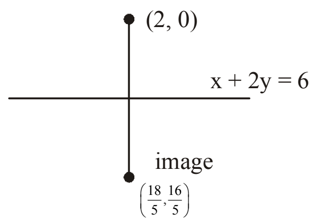
$\alpha = 2$

$\frac{\alpha+4}{2} = 3$

$\beta = 0$

Foot of directrix = (2, 0)

image of (2, 0) with respect to line $x + 2y = 6$



Directrix of image of parabola is

$x + 2y = \lambda$

$\frac{18}{5} + \frac{32}{5} = \lambda$

$\lambda = 10$

$x + 2y = 10$