JEE Main July 2021

Question Paper With Text Solution

22 July. | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JULY 2021 | 22TH JULY SHIFT-2

SECTION – A

1. Let n denote the number of solutions of the equation $z^2 + 3\overline{z} = 0$, where z is a complex number. Then the value of $\sum_{k=0}^{\infty} \frac{1}{n^k}$ is equal to :

(1) 2 (2) 4/3 (3) 1 (4) 3/2

Ans. Official Answer NTA (2)

Sol. $z^2 + 3\overline{z} = 0$

let x = x + iy

$x^2 - y^2 + 2\,ixy + 3\,(x - iy) = 0$

By comparision

$x^2 - y^2 + 3x = 0$ ...(1)

$2xy - 3y = 0$ ...(2)

$y\,(2x - 3) = 0$

$y = 0$

$x = 3/2$

If $y = 0$

then by (1) we get $x = 0, -3$

hence $x = 0, y = 0$ and $x = 0$ and $x = -3$ are two solutions.

If $x = 3/2$

then by (1) we get $y = \pm\frac{\sqrt{27}}{2}$

$x = \frac{3}{2}$, $y = \frac{\sqrt{27}}{2}$

$x = 3/2$, $y = -\frac{\sqrt{27}}{2}$ are two solution

hence total solution n = 4

$$\sum_{k=0}^{\infty} \frac{1}{4^k} = 1 + \frac{1}{4} + \frac{1}{4^2} + \ldots\ldots \infty = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

2. Let L be the line of intersection of planes $\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=2$ and $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})=2$. If P (α, β, γ) is the foot of perpendicular on L from the point (1,2, 0), then the value of 35 (α + β + γ) is equal to -

(1) 134 (2) 143 (3) 119 (4) 101

Ans. Official Answer NTA (3)

Sol. Direction of line of inter section

$$\vec{P}=\vec{n}_2\times\vec{n}_2$$

$$=\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= i\,(1-2) - j\,(-1-4) + k\,(1+2)$$

$$= -1 + 5j + 3k$$

Equation of plane in cartesicen form

$$x - y + 2z = 2 \qquad ......(1)$$

$$2x + y - z = 2 \qquad ......(2)$$

Now we will get one point on Line if intersection

then put z = 0 in (1) & (2)

$x=\frac{4}{3}$, $y=-\frac{2}{3}$

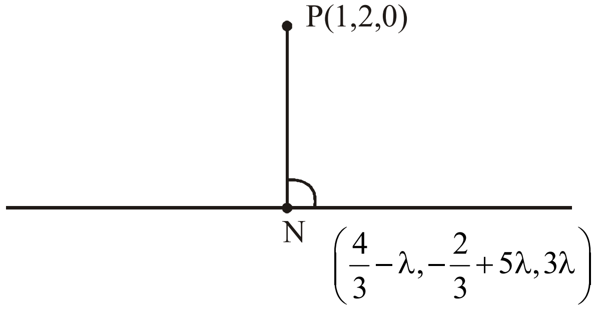
point $\left(\frac{4}{3},-\frac{2}{3},0\right)$

Now equation of L

$\vec{r}=\vec{a}+\lambda\vec{P}$

$\vec{r}=\left(\frac{4}{3}i-\frac{2}{3}j+0k\right)+\lambda\left(-i+5j+3k\right)$

We calcualte foot of perpendicular



$P\vec{N} = \left(\frac{1}{3} - \lambda\right)i + \left(5\lambda - \frac{8}{3}\right)j + (3\lambda)k$

$P\vec{N}.\vec{P} = 0$

$-1\left(\frac{1}{3} - \lambda\right) + 5\left(5\lambda - \frac{8}{3}\right) + 9\lambda = 0$

$\lambda = \frac{41}{105}$

Now $\alpha = \frac{4}{3} - \lambda$

$\beta = 5\lambda - \frac{2}{3}$

$\gamma = 3\lambda$

$\alpha + \beta + \gamma = 7\lambda + \frac{2}{3}$

$= 7\left(\frac{41}{105}\right) + \frac{2}{5} = \frac{51}{15}$

Now $35(\alpha + \beta + \gamma) = 35 \times \frac{51}{15} = 119$

3. Let [x] denote the greatest integer less than or equal to x. Then, values of $x \in R$ satisfying the equation $[e^x]^2 + [e^x + 1] - 3 = 0$ lie in the interval :

(1) [1, e] (2) $[\log_e 2, \log_e 3]$ (3) [0,1/e) (4) $[0, \log_e 2)$

Ans. Official Answer NTA (4)

Sol. $[e^x]^2 + [e^x + 1] - 3 = 0$

$[e^x]^2 + [e^x] + 1 - 3 = 0$

Let $[e^x] = y$

$y^2 + y - 2 = 0$

$(y + 2)(y - 1) = 0$

$y = -2$ Not possible

$y = 1$

$[e^x] = 1$

$1 \le e^x < 2$

By applying log

$0 \le x < \ln 2$

$x \in [0, \ln 2)$

4. The number of solutions of $\sin^7 x + \cos^7 x = 1$, $x \in [0, 4\pi]$ is equal to :

(1) 5 (2) 7 (3) 9 (4) 11

Ans. Official Answer NTA (1)

Sol. $\sin^7 x + \cos^7 x = 1 \quad : x \in [0, 4\pi]$

We know

$\sin^2 x \le 1$

$\cos^2 x \le 1$

Also

$\sin^7 x \le \sin^2 x$

$\cos^7 x \le \cos^2 x$

by adding

$\sin^7 x + \cos^7 x \le 1$ ....(1)

but $\sin^7 x + \cos^7 x = 1$ ....(2)

hence by (1) and (2)

$\sin^7 x = \sin^2 x$ ....(3)

$\cos^7 x = \cos^2 x$ ....(4)

then all solution of (3) & (4) are

$x = 0, 2\pi, 4\pi, \frac{\pi}{2}, \frac{5\pi}{2}$

total solution = 5

5. Which of the following Boolean expressions is not a tautology ?

(1) $(q \Rightarrow p) \vee (\sim q \Rightarrow p)$ (2) $(p \Rightarrow \sim q) \vee (\sim q \Rightarrow p)$

(3) $(\sim p \Rightarrow q) \vee (\sim q \Rightarrow p)$ (4) $(p \Rightarrow q) \vee (\sim q \Rightarrow p)$

Ans. Official Answer NTA (3)

Sol. Option (A) $(q \Rightarrow p) \vee (\sim q \Rightarrow p)$

$= (\sim q v p) \vee (q \, v \, p)$

$= (\sim q v q) \vee p$

$= t \vee p$

$= t$
= Which a tautology.
Option (B) $(p \Rightarrow \sim q) \vee (\sim q \Rightarrow p)$
$(\sim pv \sim q) \vee (qvp)$
$= (pv \sim p) \vee (qv \sim q)$
$= tvt$
$= t$
= Which a tautology.
Option (C) $(\sim p \Rightarrow q) \vee (\sim q \Rightarrow p)$
$(pvq) \vee (qvp)$
$= pvq$
= Which not a tautology.
Option (D) $(p \Rightarrow q) \vee (\sim q \Rightarrow p)$
$= (\sim pvq) \vee (qvp)$
$= (\sim pvp) \vee q$
$= tvq$
$= t$
= Which a tautology.

6. Four dice are thrown simultaneously and the numbers shown on these dice are recorded in 2 × 2 matrices. The probability that such formed matrices have all different entries and are non-singular, is :

(1) 23/81 (2) 45/162 (3) 22/81 (4) 43/162

Ans. Official Answer NTA (4)

Sol. Total matrices of distinct elements = ${}^6C_4 \times \underline{|4}$

Number of non singular matrices with distinct elements = ${}^6C_4 \times \underline{|4}$ – Singular matrix with different entries.

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$|X| = ad - bc = 0$

$(1, 6) \quad (3, 2)$
$(3, 4) \quad (6, 2)$

Total arrangement = 16

Hence total matrix with different entries which are singular = ${}^6C_4 \times \underline{|4} - 16 = 344$

So Required probability = $\frac{344}{6^4}$

$= \frac{43}{162}$

7. If the domain of the function $f(x) = \dfrac{\cos^{-1}\sqrt{x^2 - x + 1}}{\sqrt{\sin^{-1}\left(\dfrac{2x-1}{2}\right)}}$ is the interval (α, β], then α + β is equal to :

(1) 3/2 (2) 1/2 (3) 2 (4) 1

Ans. Official Answer NTA (1)

Sol. here

$0 \le x^2 - x + 1 \le 1$

Which is

$x^2 - x + 1 \ge 0$ and $x^2 - x + 1 \le 1$

$x \in R$ ....(1) $x^2 - x \le 0$

$x \in [0, 1]$ ....(2)

by (1) and (2)

$x \in [0, 1]$ ....(3)

Also

$$0 < \sin^{-1}\left(\frac{2x-1}{2}\right) \le \frac{\pi}{2}$$

$$0 < \frac{2x-1}{2} \le 1$$

$$0 < 2x - 1 \le 2$$

$$1 < 2x \le 3$$

$$\frac{1}{2} < x \le \frac{3}{2} \text{ ....(4)}$$

hence final answer is intersection of (3) and (4)

$$x \in \left(\frac{1}{2}, 1\right)$$

$$\alpha = \frac{1}{2}$$

$$\beta = 1$$

$$\alpha + \beta = 3/2$$

8. Let $f: R \to R$ be defined as $f(x) = \begin{cases} \dfrac{x^3}{(1-\cos 2x)} \log_e\left(\dfrac{1+2xe^{-2x}}{(1-xe^{-x})}\right) & , x \neq 0. \\ \alpha & , x = 0 \end{cases}$

If f is continuous at x = 0, then α is equal to :

(1) 0 (2) 1 (3) 2 (4) 3

Ans. Official Answer NTA (2)

Sol. If function is continuous at x = 0

RHL = f(0)

$$\lim_{h\to 0} \frac{h^3}{(1-\cos 2h)^2} \ln\left(\frac{1+2he^{-h}}{(1-he^{-h})^2}\right) = \alpha$$

$$\lim_{h\to 0} \frac{h^3}{4\sin^4 h}\left[\ln(1+2he^{-h}) - 2\ln(1-he^{-h})\right] = \alpha$$

$$\lim_{h\to 0} \frac{1}{4h}\left[\frac{\ln(1+2he^{-2h})}{2he^{-2h}}\left(2he^{-2h}\right) - \frac{2\ln(1-he^{-h})}{-he^{-h}}\left(-he^{-h}\right)\right] = \alpha$$

$$\lim_{h\to 0} \frac{1}{4h}\left[2he^{-2h} + 2he^{-h}\right] = \alpha$$

$$\lim_{h\to 0} \frac{2}{4}\left(e^{-2h} + e^{-h}\right) = \alpha$$

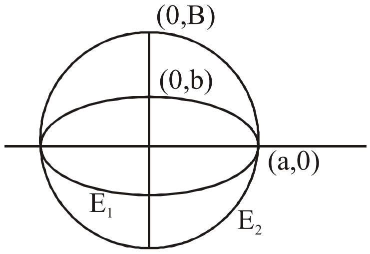
$\alpha = 1$

9. Let $E_1 : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, a > b. Let E_2 be another ellipse such that it touches the end points of major axis of E_1 and the foci of E_2 are the end points of minor axis of E_1. If E_1 and E_2 have same eccentricities, then its value is :

(1) $\dfrac{-1+\sqrt{5}}{2}$ (2) $\dfrac{-1+\sqrt{8}}{2}$ (3) $\dfrac{-1+\sqrt{6}}{2}$ (4) $\dfrac{-1+\sqrt{3}}{2}$

Ans. Official Answer NTA (1)

Sol.



E_1 :

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$e=\sqrt{1-\frac{b^2}{a^2}} \text{ ----(1)}$$

E_2:

$$\frac{x^2}{a^2}+\frac{y^2}{B^2}=1$$

but $a^2=B^2(1-e^2)$ ----(2)

also we know

b = Be

$$B=\frac{b}{e}$$

By (2)

$$a^2=\frac{b^2}{e^2}\left(1-e^2\right)$$

$$\frac{e^2}{1-e^2}=\frac{b^2}{a^2}$$

Now by (2) we get

$$e=\sqrt{1-\frac{e^2}{1-e^2}}$$

$$e^2=\frac{1-2e^2}{1-e^2}$$

$$e^2-e^4=1-2e^2$$

$$e^4-2e^2+1=e^2$$

$$\left(e^2-1\right)^2=e^2$$

$$e^2-1=\pm e$$

but $e^2-1=-e$

$$e^2+e-1=0$$

$$e=\frac{-1\pm\sqrt{5}}{2}$$

but $e=\frac{-1+\sqrt{5}}{2}$ (because e > 0)

10. If $\int_0^{100\pi} \frac{\sin^2 x}{e^{\left(\frac{x}{\pi}-\left[\frac{x}{\pi}\right]\right)}} dx = \frac{\alpha\pi^3}{1+4\pi^2}$, a ∈ R, where [x] is the greatest integer less than or equal to x, then the value of α is :

(1) $200(1-e^{-1})$ (2) $150(e^{-1}-1)$ (3) $50(e-1)$ (4) $100(1-e)$

Ans. Official Answer NTA (1)

Sol. $$\int_0^{100\pi} \frac{\sin^2 x}{e^{\left\{\frac{x}{\pi}\right\}}} dx$$

Let $f(x)=\frac{\sin^2 x}{e^{\left\{\frac{x}{\pi}\right\}}}$

Period of f(x) = LCM (π, π)= π

$$\int_0^{100\pi} \frac{\sin^2 x}{e^{\left\{\frac{x}{\pi}\right\}}} dx = 100\int_0^{\pi} \frac{\sin^2 x}{e^{\left\{\frac{x}{\pi}\right\}}} dx$$

$$=100\int_0^{\pi} \frac{\sin^2 x\, dx}{e^{\left\{\frac{x}{\pi}\right\}}}$$

x = πt
dx = πdt

$$=100\pi\int_0^{1} \frac{\sin^2 \pi t}{e^{t}}$$

$$100\pi\int_0^{1} e^{-t}\left(\frac{1-\cos 2\pi t}{2}\right)dt$$

$$=50\pi\left[\int_0^1 e^{-t}dt-\int_0^1 e^{-t}\cos 2\pi t dt\right]$$

$$=50\pi\left[\left(1-\frac{1}{e}\right)-\left(\frac{e^{-t}}{4\pi^2+1}\left(-1\cos 2\pi t+2\pi\sin 2\pi t\ \right)\right)\right]$$

$$=50\pi\left[\left(1-\frac{1}{e}\right)-\left[\frac{-e^{-1}}{1+4\pi^2}+\frac{1}{1+4\pi^2}\right]\right]$$

$= 50\pi\left[\left(1-\frac{1}{e}\right)-\frac{1}{1+4\pi^2}\left(1-\frac{1}{e}\right)\right]$

$= 50\pi\left(1-\frac{1}{e}\right)\left(\frac{4\pi^2}{1+4\pi^2}\right)$

$= \frac{200\left(1-\frac{1}{e}\right)\pi^3}{1+4\pi^2} = \frac{\alpha\pi^3}{1+4\pi^2}$

By Comparision

$\alpha = 200\,(1-e^{-1})$

11. If the shortest distance between the straight lines $3(x-1) = 6(y-2) = 2(z-1)$ and $4(x-2) = 2(y-\lambda) = (z-3)$, $\lambda \in R$ is $\frac{1}{\sqrt{38}}$, then the integral value of λ is equal to :

(1) 5 (2) – 1 (3) 2 (4) 3

Ans. Official Answer NTA (4)

Sol. Given lines

$3(x-1) = 6(y-2) = 2(z-1)$

$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-1}{3} ---(1)$

$4(x-2) = 2(y-\lambda) = 1(z-3)$

$\frac{x-2}{1} = \frac{y-\lambda}{2} = \frac{z-3}{4} ---(2)$

Minimum distance between two skew lines

$d = \left|\frac{\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$ $\quad \vec{p} = 2i + j + 3k$, $\vec{q} = i + 2j + 4k$

$A\vec{B} = i + (\lambda-2)\,j + (2)\,\hat{k}$

$\vec{p}\times\vec{q} = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = i(-2) - j(5) + k(3)$

$d = \left|\frac{-2+5(2-\lambda)+6}{\sqrt{4+25+9}}\right|$

$d = \left|\frac{-2+10-5\lambda+6}{\sqrt{38}}\right| = \frac{1}{\sqrt{38}}$

$d = \left|\frac{14-5\lambda}{\sqrt{38}}\right| = \frac{1}{\sqrt{38}}$ $= -2i - 5j + 3k$

$|5\lambda - 14| = 1$

$5\lambda - 14 = \pm 1$

$5\lambda = 14 \pm 1$

$\lambda = 3$ (integral value)

12. Let a line L : $2x + y = k$, $k > 0$ be a tangent to the hyperbola $x^2 - y^2 = 3$. If L is also a tangent to the parabola $y^2 = \alpha x$, then α is equal to :

(1) 24 (2) 12 (3) – 12 (4) –24

Ans. Official Answer NTA (4)

Sol. $x^2 - y^2 = 3$

$2x + y = k$ is tangent to hyperbola hence by condition of tangent

$c^2 = a^2 m^2 - b^2$

$y = -2x + k$ & $\frac{x^2}{3} - \frac{y^2}{3} = 1$

$k^2 = 3 \times 4 - 3$

$k^2 = 9$

$k = 3$

Now Line becomes

$y = -2x + 3$

Which is tangent to $y^2 = \alpha x$ by condition of tangently

$c = \frac{a}{m}$

$3 = \frac{\alpha}{4(-2)}$

$\alpha = -24$

13. Let the circle S: $36x^2 + 36y^2 - 108x + 120y + C = 0$ be such that it neither intersects nor touches the co-ordinates axes. If the point of intersection of the lines, $x - 2y = 4$ and $2x - y = 5$ lies inside the circles S, then :

(1) $100 < C < 156$ (2) $100 < C < 165$ (3) $81 < C < 156$ (4) $\frac{25}{9} < C < \frac{13}{3}$

Ans. Official Answer NTA (1)

Sol. $x^2 + y^2 - 3x + \frac{10}{3}y + \frac{c}{36} = 0$

We know if $x^2 + y^2 + 2gx + 2fy + c = 0$ neither touch nor intersets the co-ordinate axis then $g^2 < c$ and $f^2 < c$

By applying these condition in given circle we get

$g^2 < c$

$\frac{9}{4} < \frac{c}{36}$

$c > 81$ -----(1)

$f^2 < c$

$\frac{25}{9} < \frac{c}{36}$

$c > 100$ ----(2)

Point of intersection of $x - 2y = 4$ and $2x - y = 5$ is

$x = 2$

$y = -1$

(2, –1) lies inside circle

$s_1 < 0$

$4 + 1 - 6 - \frac{10}{3} + \frac{c}{36} < 0$

$\frac{c}{36} - \frac{13}{3} < 0$

$\frac{c}{36} < \frac{13}{3}$

$c < 156$ ----(3)

by (1), (2) and (3)

we get $100 < c < 156$

14. The values of λ and μ such that the system equations $x + y + z = 6$, $3x + 5y + 5z = 26$, $x + 2y + \lambda z = \mu$ has no solution, are :

(1) $\lambda \neq 2, \mu = 10$ (2) $\lambda = 3, \mu \neq 10$ (3) $\lambda = 2, \mu \neq 10$ (4) $\lambda = 3, \mu = 5$

Ans. Official Answer NTA (3)

Sol. here $\Delta = 0$ and at least one of Δ_x, Δ_y and Δ_z is non-zero.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 5 \\ 1 & 2 & \lambda \end{vmatrix} = 0 \quad \Rightarrow \lambda = 2$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 26 & 5 & 5 \\ \mu & 2 & \lambda \end{vmatrix} = 0 \quad \Rightarrow \mu \in R$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 3 & 26 & 5 \\ 5 & \mu & 2 \end{vmatrix} \neq 0 \quad \text{if } \mu \neq 18$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & 5 & 26 \\ 1 & 2 & \mu \end{vmatrix} \neq 0 \quad \text{if } \mu \neq 10$$

hence correct option is (3)

15. Let $y = y(x)$ be the solution of the differential equation $\operatorname{cosec}^2 x \, dy + 2dx = (1 + y\cos 2x)\operatorname{cosec}^2 x \, dx$, with $y\left(\frac{\pi}{4}\right) = 0$. Then, the value of $(y(0) + 1)^2$ is equal to -

(1) e^{-1} (2) e (3) $e^{\frac{-1}{2}}$ (4) $e^{\frac{1}{2}}$

Ans. Official Answer NTA (1)

Sol. $\operatorname{cosec}^2 x \, dy + 2dx = (1 + y\cos 2x)\operatorname{cosec}^2 x$

$$\frac{dy}{dx} + 2\sin^2 x = 1 + y\cos 2x$$

$$\frac{dy}{dx} + y(-\cos 2x) = 1 - 2\sin^2 x$$

$$\frac{dy}{dx} + y(-\cos 2x) = \cos 2x$$

$$\frac{dy}{dx}+Py=Q$$

I.F. $= e^{\int -\cos 2x\, dx}$

$= e^{-\frac{1}{2}\sin 2x}$

y(I.F.) $= \int (\text{I.F.})\, Q dx$

$$y\left(e^{-\frac{1}{2}\sin 2x}\right)=\int e^{-\frac{1}{2}\sin 2x}\cos 2x\, dx$$

$$-\frac{1}{2}\sin 2x = t$$

$$-\cos 2x\, dx = dt$$

$$\cos 2x\, dx = -dt$$

$$y\left(e^{-\frac{1}{2}\sin 2x}\right)=\int e^{t}(-dt)$$

$$ye^{-\frac{1}{2}\sin 2x} = -e^{t}+C$$

$$ye^{-\frac{1}{2}\sin 2x} = -e^{-\frac{1}{2}\sin 2x}+C$$

by $y\left(\frac{\pi}{4}\right)=0$

$$0=-e^{-\frac{1}{2}}+C$$

$$C=\frac{1}{\sqrt{e}}$$

$$ye^{-\frac{1}{2}\sin 2x} = -e^{-\frac{1}{2}\sin 2x}+\frac{1}{\sqrt{C}}$$

at $x = 0$

$$y(0)=-1+\frac{1}{\sqrt{e}}$$

$$\left(y(0)+1\right)^2=\left(-1+\frac{1}{\sqrt{e}}+1\right)^2=\left(e^{-\frac{1}{2}}\right)^2$$

$$=e^{-1}$$

16. Let A = [a_{ij}] be a real matrix of order 3 × 3, such that $a_{i1} + a_{i2} + a_{i3} = 1$, for i = 1,2,3. Then, the sum of all the entries of the matrix A^3 is equal to :

(1) 9 (2) 1 (3) 2 (4) 3

Ans. Official Answer NTA (4)

Sol.
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{Let } X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A\,X = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad A\,X = X$

Replace X by A X

$A\,(A\,X) = A\,X$

$A^2\,X = A\,X = X$

$A^2\,X = X$

Replace X by A X

$A^3\,X = A\,X = X$

$$\text{Let } A^3 \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Now

$$A^3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ y_1 + y_2 + y_3 \\ z_1 + z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

hence sum of elements = 3

17. Let a vector $\vec{a}$ be coplanar with vectors $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} + \hat{k}$. If $\vec{a}$ is perpendicular to $\vec{d} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, and $|\vec{a}| = \sqrt{10}$. Then possible value of $\left[\vec{a}\ \vec{b}\ \vec{c}\right] + \left[\vec{a}\ \vec{b}\ \vec{d}\right] + \left[\vec{a}\ \vec{c}\ \vec{d}\right]$ is equal to :

(1) – 40 (2) – 38 (3) – 42 (4) –29

Ans. Official Answer NTA (3)

Sol. $\vec{a} = \lambda\vec{b} + \mu\vec{c}$

$\vec{a} = \lambda(2i + j + k) + \mu(i - j + k)$

$\vec{a} = i(2\lambda + \mu) + j(\lambda - \mu) + k(\lambda + \mu)$

Now $\vec{a}.\vec{d} = 0$

$3(2\lambda + \mu) + 2(\lambda - \mu) + 6(\lambda + \mu) = 0$

$14\lambda + 7\mu = 0$

$2\lambda + \mu = 0$ ....(1)

$|\vec{a}| = \sqrt{10}$

$\sqrt{(2\lambda + \mu)^2 + (\lambda - \mu)^2 + (\lambda + \mu)^2} = \sqrt{10}$

$4\lambda^2 + 4\lambda\mu + \mu^2 + 2\lambda^2 + 2\mu^2 = 10$

$6\lambda^2 + 3\mu^2 + 4\lambda\mu = 10$ ....(2)

by (1) $\mu = -2\lambda$

$6\lambda^2 + 12\lambda^2 - 8\lambda^2 = 10$

$10\lambda^2 = 10$

$\lambda = \pm 1$

if $\lambda = 1$ then $\mu = -2$

if $\lambda = -1$ then $\mu = 2$

Now $\vec{a} = 3j - k$

or

$\vec{a} = -3j + k$

if we choose $\vec{a} = 3j - k$

$\left[\vec{a}\,\vec{b}\,\vec{c}\right] + \left[\vec{a}\,\vec{b}\,\vec{d}\right] + \left[\vec{a}\,\vec{c}\,\vec{d}\right] = 0 - 28 - 14 = -42$

18. Let three vector $\vec{a}$, $\vec{b}$ and $\vec{c}$ be such that $\vec{a}\times\vec{b}=\vec{c}$, $\vec{b}\times\vec{c}=\vec{a}$ and $|\vec{a}|=2$. Then which one of the following is not true ?

(1) Projection of $\vec{a}$ on $\left(\vec{b}\times\vec{c}\right)$ is 2 (2) $\left|3\vec{a}+\vec{b}-2\vec{c}\right|^2=51$

(3) $\vec{a}\times\left(\left(\vec{b}+\vec{c}\right)\times\left(\vec{b}-\vec{c}\right)\right)=\vec{0}$ (4) $\left[\vec{a}\ \vec{b}\ \vec{c}\right]+\left[\vec{c}\ \vec{a}\ \vec{b}\right]=8$

Ans. Official Answer NTA (2)

Sol. $\vec{a}\times\vec{b}=\vec{c}$ .....(1)

$\vec{b}\times\vec{c}=\vec{a}$ .....(2)

and $|\vec{a}|=2$.

By (1) and (2)

$\vec{a}.\vec{c}=0$

$\vec{b}.\vec{c}=0$

$\vec{a}.\vec{b}=0$

Hence $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular

$\vec{a}\times\vec{b}=\vec{c}$

$|\vec{a}\times\vec{b}|=|\vec{c}|$

$a^2b^2\sin^2 90^\circ=c^2$

$4b^2=c^2$ ....(3)

$\vec{b}\times\vec{c}=\vec{a}$

$|\vec{b}\times\vec{c}=\vec{a}|$

$b^2c^2\sin^2 90^\circ=a^2$

$b^2c^2=4$ ....(4)

by (3) & (4)

$|\vec{b}|=1$

$|\vec{c}|=2$

Now

(1) option (A)

$$\left|\frac{\vec{a}.(\vec{b}\times\vec{c})}{\vec{b}\times\vec{c}}\right|=2$$

$$\frac{|\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\theta}{|\vec{b}||\vec{c}|\sin\theta}=2$$

2 = 2 which is correct.

(B) $\left|3\vec{a}+\vec{b}-2\vec{c}\right|^2 = 9a^2+b^2+4c^2+0+0+0$

$= 36+1+16$

Which is wrong

hence option (2) is correct

19. Let S_n denote the sum of first n-terms of an arithmetic progression. If $S_{10} = 530$, $S_5 = 140$, then $S_{20} - S_6$ is equal to :

(1) 1862 (2) 1872 (3) 1852 (4) 1842

Ans. Official Answer NTA (1)

Sol. $S_{10} = 530$

$\frac{10}{2}(2a+9d) = 530$

$2a + 9d = 106$ ....(1)

$S_5 = \frac{5}{2}(2a+4d) = 140$

$a + 2d = 28$ ....(2)

By (1) and (2)

$d = 10$, $a = 8$

Now $S_{20} - S_6 = \frac{20}{2}[2a+19d] - \frac{6}{2}[2a+5d]$

$= 1862$

20. Let $f : R \to R$ be defined as $f(x) = \begin{cases} -\frac{4}{3}x^3 + 2x^2 + 3x & , x > 0 \\ 3xe^x & , x \le 0 \end{cases}$. Then f is increasing function in the interval.

(1) $\left(-1, \frac{3}{2}\right)$ (2) (0, 2) (3) (–3, –1) (4) $\left(-\frac{1}{2}, 2\right)$

Ans. Official Answer NTA (1)

Sol. $f'(x) = \begin{cases} -4x^2 + 4x + 3 & ; x > 0 \\ 3e^x + 3x\,e^x & ; x \le 0 \end{cases}$

If function is increasing then

$f'(x) > 0$

$-4x^2 + 4x + 3 > 0$

$4x^2 - 4x - 3 \le 0$

$4x^2 - 6x + 2x - 3 < 0$

$2x(2x - 3) + 1(2x - 3) \le 0$

$x \in \left(-\frac{1}{2}, \frac{3}{2}\right)$ but here $x > 0$

then $x \in \left(0, \frac{3}{2}\right)$ ....(1)

$3e^x + 3x\,e^x > 0$
$x + 1 > 0$
$x > -1$ ....(2)
by (1) and (2)

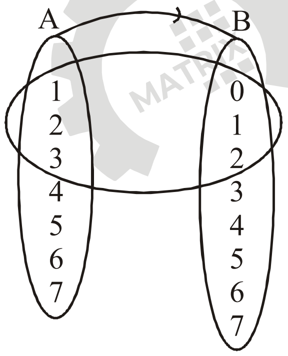
$x \in \left(-1, \frac{3}{2}\right)$

SECTION - B

1. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Then the number of bijective function $f : A \rightarrow A$ such that $f(1) + f(2) = 3 - f(3)$ is equal to ___.

Ans. Official Answer NTA (720)

Sol.



1, 2, 3 will be associated with 0, 1, 2 since $f(1) + f(2) + f(3) = 3$, then no. of method for these = $\lfloor 3$

Now 0, 4, 5, 6, 7 will be associated with 3, 4, 5, 6, 7 then no. of methods = $\lfloor 5$

total method = $\lfloor 5 \times \lfloor 3$

= 720

2. The number of elements in the set $\{n \in \{1,2,3,........100\} | (11)^n > (10)^n + (9)^n\}$ is ____.

Ans. Official Answer NTA (96)

Sol. $11^n > 10^n + 9^n$

$\Rightarrow 11^n - 9^n > 10^n$

$\Rightarrow (10+1)^n - (10-1)^n > 10^n$

$\Rightarrow 2\left(n_{C_1} 10^{n-1} + n_{C_3} 10^{n-3} + n_{C_5} 10^{n-5} + ...\right) > 10^n$

$\Rightarrow \frac{1}{5}\left(n_{C_1} 10^{n} + n_{C_3} 10^{n-2} + n_{C_5} 10^{n-4} + ...\right) > 10^n$

$\Rightarrow \frac{1}{5}\left(n_{C_1} + n_{C_3} 10^{-2} + n_{C_5} 10^{-4} + ...\right) > 1$

$\Rightarrow$ Clearly this is true for $n \geq 5$

hence such numbers in given interval are 96.

3. The sum of all the elements in the set $\{n \in \{1,2........, 100\} |$ H.C.F. of n and 2040 is 1$\}$ is equal to ______.

Ans. Official Answer NTA (1251)

Sol. $2040 = 2^4 \times 3^1 \times 5^1 \times 17^1$

If H.C.F is 'l' then 'n'should not be multiple of 2, 3, 5, 17

= (1 + 2 + 3 + 100)– (2 + 4 + 6.....100)– (3 + 6 + 9..) – (5 + 10 + 15.. – (17 + 34 + 51 .....)

= 1251

4. The area (in sq. units) of the region bounded by the curves $x^2 + 2y - 1 = 0$, $y^2 + 4x - 4 = 0$ and $y^2 - 4x - 4 = 0$, in the upper half plane is.

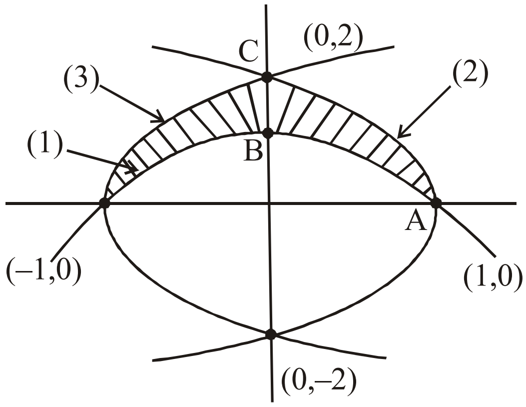
Ans. Official Answer NTA (2)

Sol. $x^2 = -2y + 1$

$x^2 = -2(y - \frac{1}{2})$ ......(1)

$y^2 = -4(x - 1)$ ......(2)

$y^2 = 4(x + 1)$ ......(3)



Required area = 2 (area ABC)

$$=2\int_0^1\left(2\sqrt{1-x}-\frac{\left(1-x^2\right)}{2}\right)dx$$

= 2 square unit.

5. If the digits are not allowed to repeat in any number formed by using the digits 0, 2, 4, 6, 8, then the number of all numbers greater than 10,000 is equal to.

Ans. Official Answer NTA (96)

Sol.

4 ways	4 ways	3 ways	2 ways
↑	↑	↑	↑

$4 \times 4 \times 3 \times 2 = 96$

6. If the constant term, in binomial expansion of $\left(2x^r+\frac{1}{x^2}\right)^{10}$ is 180, then r is equal to.

Ans. Official Answer NTA (8)

Sol. Let, T_{n+1} be the constant term

$T_{n+1} = 180$

$${}^{10}C_n\left(2x^r\right)^n\left(\frac{1}{x^2}\right)^{10-n}=180$$

$${}^{10}C_n\left(2^n\right)\left(x^{rn-20+2n}\right)=180$$

Now by comparision

${}^{10}C_n\, 2^n = 180$ .....(1)

$2n + rn - 20 = 0$ .....(2)

$n = 2$

$r = 8$

7. Consider the following frequency distribution :

Class :	0 –6	6 –12	12– 18	18 – 24	24 – 30
Frequency :	a	b	12	9	5

If mean = $\frac{309}{22}$ and median = 14, then the value $(a-b)^2$ is equal to.

Official Answer NTA (4)

Sol.

Class	x_i	f_i	$f_i x_i$
0 – 6	3	a	3a
6 – 12	9	b	9b
12 – 18	15	12	180
18 – 24	21	9	189
24 – 30	27	5	135

Mean = $\frac{3a+9b+180+189+135}{a+b+12+9+5} = \frac{309}{22}$

$18a + 37b = 1018$ ....(1)

Now median $= 12 + \left(\frac{\frac{a+b+26}{2} - \ a+b}{12}\right) \times 6 = 14$

$a + b = 18$ ....(2)

by (1) & (2)

$a = 8, \quad b = 10$

$(a-b)^2 = 4$

8. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then the number of 3 × 3 matrices B with entries from the set {1, 2, 3, 4, 5} and satisfying AB = BA is.

Ans. Official Answer NTA (3125)

Sol. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $B=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

given AB = BA

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix}=\begin{bmatrix} a_2 & a_1 & a_3 \\ b_2 & b_1 & b_3 \\ c_2 & c_1 & c_3 \end{bmatrix}$$

By comparing both sides

$b_1 = a_2$ $\quad$ $c_1 = c_2$

$b_2 = a_1$ $\quad$ $c_3 = c_3$

$b_3 = a_3$

Now b_1, b_2, b_3 $\quad$ c_1, c_2, c_3 can be choosen by = 5 × 5 = 25 ways

a_1, a_2, a_3 can be choosen by

= 5 × 5 × 5 = 125 way

total number of ways = 125 × 25 = 3125 ways

9. Let y = y (x) be solution of the differential equation $\left((x+2)e^{\left(\frac{y+1}{x+2}\right)}+(y+1)\right)$ dx = (x + 2)dy, y(1) = 1. If the domain of y = y(x) is an open interval (α,β), then |α + β| is equal to.

Official Answer NTA (4)

Sol. $$\left((x+2)e^{\frac{y+1}{x+2}}+(y+1)\right)dx=(x+2)dy$$

$x+2=X \quad \Rightarrow \quad dx=dX$

$y+1=Y \quad \Rightarrow \quad dy=dY$

$$\left(Xe^{\frac{Y}{X}}+Y\right)dX=X\,dY$$

$$\frac{dY}{dX}=e^{\frac{Y}{X}}+\frac{Y}{X}$$

put $Y = tX$

$$\frac{dY}{dx} = t + X\frac{dt}{dX}$$

$$\Rightarrow t + X\frac{dt}{dX} = e^t + t$$

$$\Rightarrow e^{-t}dt = \frac{dX}{X}$$

$$\Rightarrow -e^{-t} = \ln|X| + C$$

$$\Rightarrow -e^{-\frac{Y}{X}} = \ln|X| + C$$

Now this pass through (3, 2)

$$-e^{-\frac{2}{3}} = \ln 3 + C$$

then

$$-e^{-\frac{Y}{X}} = \ln|X| - e^{-\frac{2}{3}} - \ln 3$$

$$-e^{-\frac{Y}{X}} = \ln 3 + e^{-\frac{2}{3}} - \ln|X| > 0$$

$$\ln|X| < e^{-\frac{2}{3}} + \ln 3$$

let $\lambda = e^{-\frac{2}{3}} + \ln 3$

$\ln|X| < \lambda$

$|X| < e^{\lambda}$

$|x+2| < e^{\lambda}$

$-e^{\lambda} < x + 2 < e^{\lambda}$

$-e^{\lambda} - 2 < x < e^{\lambda} - 2$

$\alpha = -e^{\lambda} - 2$

$\beta = e^{\lambda} - 2$

$|\alpha + \beta| = 4$

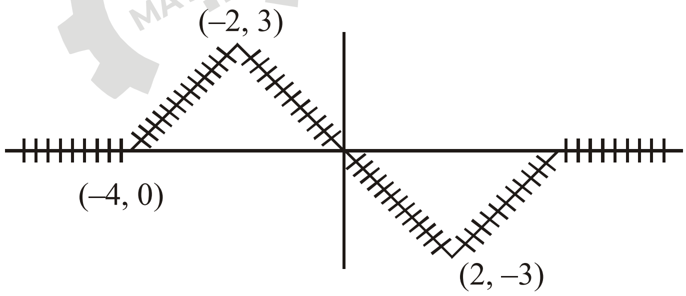
10. Let $f : R \to R$ be a function defined as $f(x) = \begin{cases} 3\left(1 - \frac{|x|}{2}\right) & \text{if } |x| \le 2 \\ 0 & \text{if } |x| > 2 \end{cases}$. Let $g : R \to R$ be given by $g(x) = f(x + 2) - f(x - 2)$. If n and m denote the number of points in R where g is not continuous and not differentiable, respectively, then n + m is equal to.

Official Answer NTA (4)

Sol. $f(x-2) = \begin{cases} \dfrac{3x}{2} & : \quad -4 \le x \le -2 \\ \dfrac{-3x}{2} & : \quad -2 < x \le 0 \\ 0 & : \quad x \in (-\infty, -4) \cup (0, \infty) \end{cases}$

$f(x-2) = \begin{cases} \dfrac{3x}{2} & : \quad 0 \le x \le 2 \\ \dfrac{-3x}{2} + 6 & : \quad 2 \le x \le 4 \\ 0 & : \quad x \in (-\infty, 0) \cup (4, \infty) \end{cases}$

$g(x) = f(x+2) - f(x-2) = \begin{cases} \dfrac{3x}{2} + 6 & : \quad -4 \le x \le -2 \\ \dfrac{-3x}{2} & : \quad -2 < x < 2 \\ \dfrac{3x}{2} - 6 & : \quad 2 \le x \le 4 \\ 0 & : \quad x \in (-\infty, -4) \cup (4, \infty) \end{cases}$



here n = 0

m = 4

hence m + n = 4