JEE Main July 2021

Question Paper With Text Solution

20 July. | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN JULY 2021 | 20TH JULY SHIFT-2

SECTION – A

1. In a triangle ABC, if $|\overrightarrow{BC}| = 3, |\overrightarrow{CA}| = 5$ and $|\overrightarrow{BA}| = 7,$ then the projection of the vector $\overrightarrow{BA}$ **on** $\overrightarrow{BC}$ **is equal to**

(1) $\frac{11}{2}$

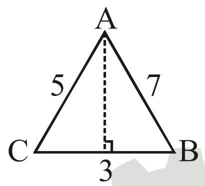
(2) $\frac{19}{2}$

(3) $\frac{13}{2}$

(4) $\frac{15}{2}$

Ans. Official Answer NTA (1)

Sol. $|\overrightarrow{BC}| = 3, |\overrightarrow{CA}| = 5, |\overrightarrow{BA}| = 7$



Projection of $\overrightarrow{BA}$ on $\overrightarrow{BC} = |\overrightarrow{BA}|$ Cos $\angle ABC$

$$= 7\left(\frac{3^2 + 7^2 - 5^2}{2 \times 3 \times 7}\right) = \frac{11}{2}$$

2. If $[x]$ denotes the greatest integer less than or equal to x, then the value of the integral $\int_{-\pi/2}^{\pi/2} [[x] - \sin x] dx$ is equal to :

(1) 0

(2) π

(3) $-\pi$

(4) 1

Ans. Official Answer NTA(3)

Sol. $I = \int_{-\pi/2}^{\pi/2} ([x] + [-\sin x]) dx \; \{\because [x + I] = [x] + I\}$

Using property $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx$

$$I = \int_{0}^{\frac{\pi}{2}} ([x]+[-\sin x])dx + \int_{0}^{\frac{\pi}{2}} ([-x]+[-\sin x])dx$$

$$I = \int_{0}^{\pi/2} ([x]+[-x])\,dx + \int_{0}^{\pi/2} ([\sin x]+[-\sin x])\,dx$$

$$I = -\int_{0}^{\pi/2} dx - \int_{0}^{\pi/2} dx$$

$I = -\pi$

3. Let r_1 and r_2 be the radii of the largest and smallest circles respectively, which pass through the point (–4, 1) and having their centres on the circumference of the circle $x^2 + y^2 + 2x + 4y - 4 = 0$. If $\frac{r_1}{r_2} = a + b\sqrt{2}$, then a + b is equal to :

(1) 11

(2) 7

(3) 5

(4) 3

Ans. Official Answer NTA (3)

Sol. $(x+1)^2 + (y+1)^2 = 9$

Let centre of circles is (–1 + 3 cosθ, –2 + 3 sinθ,)

r = distance between centre and point (–4, 1)

$$r = \sqrt{(3+3\cos\theta)^2 + (-3+3\sin\theta)^2}$$

$$r = 3\sqrt{3+2(\cos\theta - \sin\theta)}$$

$$r_1 = r_{max} = 3\sqrt{3+2\sqrt{2}}$$

$$r_2 = r_{min} = 3\sqrt{3-2\sqrt{2}}$$

$$\frac{r_1}{r_2} = \frac{3\sqrt{3+2\sqrt{2}}}{3\sqrt{3-2\sqrt{2}}} = 3 + 2\sqrt{2}$$

a = 3, b = 2

a + b = 5

4. Let P be a variable point on the parabola $y = 4x^2 + 1$. Then the locus of the mid-point of the point P and the foot of the perpendicular drawn from the point P to the line y = x is:

(1) $2(3x - y)^2 + (x - 3y) + 2 = 0$

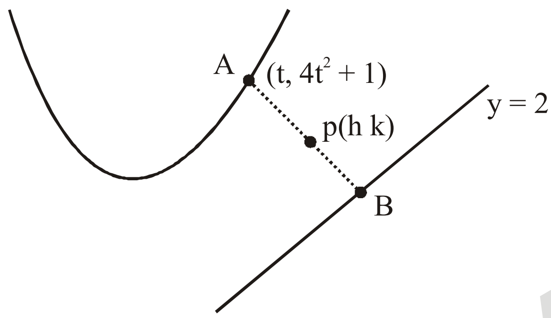
(2) $(3x - y)^2 + 2(x - 3y) + 2 = 0$

(3) $(3x - y)^2 + (x - 3y) + 2 = 0$

(4) $2(x - 3y)^2 + (3x - y) + 2 = 0$

Ans. Official Answer NTA (1)

Sol.



equation of AB

$y - (4t^2 + 1) = -1(x - t)$

$B\left(\frac{4t^2 + t + 1}{2}\ \ \frac{4t^2 + t + 1}{2}\right)$

$h = \dfrac{\frac{4t^2 + t + 1}{2} + t}{2}$, $\qquad k = \dfrac{\frac{4t^2 + t + 1}{2} + 4t^2 + 1}{2}$,

$4h = 4t^2 + 3t + 1$ ....(1)

$4k = 12t^2 + t + 3$ ....(2)

subtract both equation

$t = \dfrac{k - 3h}{2}$ put in (1)

we get $2(3h - k)^2 + (h - 3k) + 2 = 0$

Replace h by x and k by y

$2(3x - y)^2 + (x - 3y) + 2 = 0$

5. The value of $k \in R$, for which the following system of linear equations

$3x - y + 4z = 3,$

$x + 2y - 3z = -2,$

$6x + 5y + kz = -3,$

has infinitely many solutions, is :

(1) 3 (2) –5 (3) –3 (4) 5

Ans. Official Answer NTA (2)

Sol. For infinitely many solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$$\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & k \end{vmatrix} = 0$$

$3(2k + 15) + 1(k + 18) + 4(5 - 12) = 0$

$6k + 45 + k + 18 - 28 = 0$

$7k = -35$

$k = -5$

at $k = -5$

$\Delta_x = \Delta_y = \Delta_z = 0$

6. If sum of the first 21 terms of the series $\log_{9^{\frac{1}{2}}} x + \log_{9^{\frac{1}{3}}} x + \log_{9^{\frac{1}{4}}} x + \ldots,$ where $x > 0$ is 504, then x is equal to :

(1) 243

(2) 7

(3) 9

(4) 81

Ans. Official Answer NTA (4)

Sol. $2\log_9^x + 3\log_9^x + 4\log_9^x + \ldots\ldots\ldots\ldots$ 21 terms

$2\log_9^x (2 + 3 + 4 + \ldots\ldots\ldots + 22) = 504$

$252 \log_9^x = 504$

$\log_9^x = 2$

$x = 81$

7. Let $f : R - \left\{\frac{\alpha}{6}\right\} \to R$ be defined by $f(x) = \frac{5x+3}{6x-\alpha}$. Then the value of α for which (f0f) (x) = x, for all $x \in R - \left\{\frac{\alpha}{6}\right\}$, is :

(1) 8

(2) 5

(3) 6

(4) No such α exists

Ans. Official Answer NTA (2)

Sol. $f\left(f(x)\right) = x$

$$\frac{5f(x)+3}{6f(x)-2} = x$$

$5\ f(x) + 3 = 6x\ f(x) - x\ \alpha$

$$f(x) = \frac{-3-x\alpha}{5-6x} = \frac{x\alpha+3}{6x-5}$$

$$\frac{5x+3}{6x-\alpha} = \frac{x\alpha+3}{6x-5}$$

$30x^2 - 25x + 18x - 15 = 18x + 6\alpha\, x^2 - 3\alpha - x\alpha^2$

$6\alpha = 30$

$\alpha = 5$

8. Consider the following three statements :

(A) If 3 + 3 = 7 then 4 + 3 = 8.

(B) If 5 + 3 = 8 then earth is flat.

(C) If both (A) and (B) are true then 5 + 6 = 17.

Then, which of the following statements is correct ?

(1) (A) is false, but (B) and (C) are true

(2) (A) is true while (B) and (C) are false

(3) (A) and (C) are true while (B) is false

(4) (A) and (B) are false while (C) is true

Ans. Official Answer NTA (3)

Sol. Truth table $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	F
F	F	T

A is true, B is false, C is true

9. Let $g(t)=\int_{-\pi/2}^{\pi/2} \cos\left(\frac{\pi}{4}t+f(x)\right)dx$, where $f(x)=\log_e(x+\sqrt{x^2+1}), x \in R$. Then which one of the following is correct ?

(1) $g(1)=g(0)$

(2) $\sqrt{2}\,g(1)=g(0)$

(3) $g(1)+g(0)=0$

(4) $g(1)=\sqrt{2}\,g(0)$

Ans. Official Answer NTA (2)

Sol. $g(1)=\int\limits_{-\pi/2}^{\pi/2}\cos\left(\frac{\pi}{4}+f(x)\right)dx$

$g(1)=\int\limits_{-\pi/2}^{\pi/2}\cos\frac{\pi}{4}\cos(f(x))dx-\int\limits_{-\pi/2}^{\pi/2}\sin\frac{\pi}{4}\sin(f(x))dx$

$g(1)=\frac{1}{\sqrt{2}}\int\limits_{-\pi/4}^{\pi/4}\cos(f(x))dx-\frac{1}{\sqrt{2}}\int\limits_{-\pi/2}^{\pi/2}\sin\left(\log\left(n+\sqrt{x^2+1}\right)\right)dx$

$\left\{\text{we know that } \sin\left(\log_e\left(4+\sqrt{x^2+1}\right)\right) \text{is a odd funtion}\right\}$

$g(1)=\frac{1}{\sqrt{2}}\int\limits_{-\pi/4}^{\pi/4}\cos(f(x))dx+0$

$\sqrt{2}g(1)=g(0)$

10. The sum of all the local minimum values of the twice differentiable function $f : R \to R$ defined by $f(x) = x^3 - 3x^2 - \frac{3f''(2)}{2}x + f''(1)$ is :

(1) – 27

(2) 0

(3) 5

(4) – 22

Ans. Official Answer NTA (1)

Sol. $f(x) = x^3 - 3x^2 - \frac{3}{2}f''(2)x + f''(1)$

$f'(x) = 3x^2 - 6x - \frac{3}{2}f''(2)$

$f''(x) = 6x - 6$

$f''(1) = 0, \quad f''(2) = 6$

$f(x) = x^3 - 3x^2 - 9x$

$f'(x) = 0$

$3x^2 - 6x - 9 = 0$

$x = -1$ and $x = 3$

Locul minima at x = 3

$f(3) = 27 - 27 - 27 = -27$

11. Let in a right angled triangle, the smallest angle be θ. If a triangle formed by taking the reciprocal of its sides is also a right angled triangle, then $\sin\theta$ is equal to :

(1) $\frac{\sqrt{2}-1}{2}$

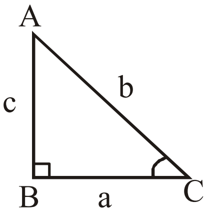
(2) $\frac{\sqrt{5}-1}{4}$

(3) $\frac{\sqrt{5}+1}{4}$

(4) $\frac{\sqrt{5}-1}{2}$

Ans. Official Answer NTA (4)

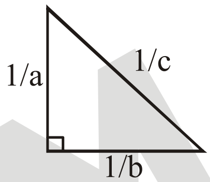
Sol.



$a^2 + c^2 = b^2$ ....(1)

$\sin\theta = \frac{c}{b}$

smallest angle is 'θ'
so smallest side is 'c'
$b > a > c$



$\frac{1}{b} < \frac{1}{a} < \frac{1}{c}$

$\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$

$\frac{b^2}{c^2} = \frac{b^2}{a^2} + 1$ ....(2)

from $\frac{a^2}{b^2} + \frac{c^2}{b^2} = 1$ ....(3)

from (2) and (3)

$\frac{a^2}{b^2} = 1 - \frac{c^2}{b^2} = \frac{1}{\frac{b^2}{c^2} - 1}$

put $\frac{c^2}{b^2} = t$

$1 - t = \frac{1}{\frac{1}{t} - 1}$

$(1 - t)^2 = t$

$t^2 - 3t + 1 = 0$

$t = \dfrac{3 \pm \sqrt{5}}{2}$

$\dfrac{c^2}{b^2} = t = \dfrac{3+\sqrt{5}}{2}$

$\sin^2\theta = \dfrac{3+\sqrt{5}}{2}$

$\sin\theta = \sqrt{\dfrac{3+\sqrt{5}}{2}}$

$\sin\theta = \sqrt{\dfrac{6-2\sqrt{5}}{4}}$

$\sin\theta = \dfrac{\sqrt{5}-1}{2}$

12. Let A, B and C be three events such that the probability that exactly one of A and B occurs is $(1 - k)$, the probability that exactly one of B and C occurs is $(1 - 2k)$, the probability that exactly one of C and A occurs is $(1 - k)$ and the probability of all A, B and C occur simultaneously is k^2, where $0 < k < 1$. Then the probability that at least one of A, B and C occur is :

(1) greater than $\dfrac{1}{2}$

(2) greater than $\dfrac{1}{8}$ but less than $\dfrac{1}{4}$

(3) exactly equal to $\dfrac{1}{2}$

(4) greater than $\dfrac{1}{4}$ but less than $\dfrac{1}{2}$

Ans. Official Answer NTA (1)

Sol. Probability of exactly one of

A and B occurs $= P(A) + P(B) - 2P(A \cap B)$

$P(A) + P(B) - 2P(A \cap B) = 1 - K$

$P(B) + P(C) - 2P(B \cap C) = 1 - 2K$

$P(C) + P(A) - 2P(A \cap C) = 1 - K$

$2(P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) = 3 - 4K$ and $P(A \cap B \cap C) = K^2$ ....(1)

Probability of at least one of A, B or C occur = $P(A \cup B \cup C)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = \frac{3-4k}{2} + k^2 \quad = k^2 - 2k + \frac{3}{2}$

$= (k-1)^2 + \frac{1}{2}$

$P(A \cup B \cup C) \geq \frac{1}{2}$

13. Let $y = y(x)$ satisfies the equation $\frac{dy}{dx} - |A| = 0,$ for all $x > 0$, where $A = \begin{bmatrix} y & \sin x & 1 \\ 0 & -1 & 1 \\ 2 & 0 & \frac{1}{x} \end{bmatrix}$.

If $y(\pi) = \pi + 2,$ then the value of $y\left(\frac{\pi}{2}\right)$ is :

(1) $\frac{\pi}{2} + \frac{4}{\pi}$

(2) $\frac{3\pi}{2} - \frac{1}{\pi}$

(3) $\frac{\pi}{2} - \frac{4}{\pi}$

(4) $\frac{\pi}{2} - \frac{1}{\pi}$

Ans. Official Answer NTA (1)

Sol. $\frac{dy}{dx} = |A| = \begin{vmatrix} y & \sin x & 1 \\ 0 & -1 & 1 \\ 2 & 0 & \frac{1}{x} \end{vmatrix}$

expand along R_3

$\frac{dy}{dx} = 2(\sin x + 1) + \frac{1}{x}(-y)$

$$\frac{dy}{dx} = 2(\sin x + 1) - \frac{y}{x}$$

$$\frac{dy}{dx} = \left(\frac{1}{x}\right)y = 2(\sin x + 1)$$

$$\text{I.F.} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x\frac{dy}{dx} + y = 2x(\sin x + 1)$$

$$d(xy) = \int 2x(\sin x + 1)dx$$

$$\frac{xy}{2} = -x\cos x - \int(-\cos x)dx + \frac{x^2}{2} + C$$

$$\frac{xy}{2} = -x\cos x + \sin x + \frac{x^2}{2} + C$$

$$\frac{\pi(\pi+2)}{2} = -\pi(-1) + 0 + \frac{\pi^2}{2} + C$$

$C = 0$

$$\frac{\frac{\pi}{2}y\left(\frac{\pi}{2}\right)}{2} = 0 + 1 + \frac{\pi^2}{8} \quad \Rightarrow y$$

$$y\left(\frac{\pi}{2}\right) = \frac{4}{\pi} + \frac{\pi}{2}$$

14. The lines $x = ay - 1 = z - 2$ and $x = 3y - 2 = bz - 2, (ab \neq 0)$ are coplanar, if :

(1) $b = 1, a \in R - \{0\}$

(2) $a = 2, b = 2$

(3) $a = 2, b = 3$

(4) $a = 1, b \in R - \{0\}$

Ans. Official Answer NTA (1)

Sol. $$\frac{x-0}{1} = \frac{y-\frac{1}{a}}{1/a} = \frac{z-2}{1}$$

$$\frac{x}{1} = \frac{y-\frac{2}{3}}{\frac{1}{3}} = \frac{z-\frac{2}{b}}{\frac{1}{b}}$$

if lines are coplaner then

$$\begin{vmatrix} 0 & \frac{2}{3}-\frac{1}{a} & \frac{2}{b}-2 \\ 1 & \frac{1}{a} & 1 \\ 1 & \frac{1}{3} & \frac{1}{b} \end{vmatrix} = 0$$

3b – 4a + 4ab = 3

from option (1)

$b = 1$ $a \in R - \{0\}$

15. The value of $\tan\left(2\tan^{-1}\left(\frac{3}{5}\right)+\sin^{-1}\left(\frac{5}{13}\right)\right)$ is equal to :

(1) $\frac{151}{63}$

(2) $\frac{-291}{76}$

(3) $\frac{-181}{69}$

(4) $\frac{220}{21}$

Ans. Official Answer NTA (4)

Sol. $2\tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\frac{\frac{3}{5}+\frac{3}{5}}{1-\frac{9}{25}}$

$= \tan^{-1}\left(\frac{15}{8}\right)$

$= \tan\left(\tan^{-1}\frac{15}{8}+\tan^{-1}\frac{5}{12}\right)$

$= \tan\left(\tan^{-1}\frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{75}{96}}\right)$

$=\frac{220}{21}$

16. If the real part of the complex number $(1-\cos\theta+2i\sin\theta)^{-1}$ is $\frac{1}{5}$ for $\theta\in(0,\pi)$, then the value of the integral $\int_0^{\theta} \sin x\, dx$ is equal to :

(1) 1

(2) 0

(3) – 1

(4) 2

Ans. Official Answer NTA (1)

Sol. $$\frac{1}{2\sin^2\frac{\theta}{2}+4i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\frac{1}{2\sin\frac{\theta}{2}}\left(\frac{\sin\frac{\theta}{2}-2i\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+4\cos^2\frac{\theta}{2}}\right)$$

Real part = $2^{\frac{1}{\left(1+3\cos^2\frac{\theta}{2}\right)}}=\frac{1}{5}$

$\cos\theta = 0$

$\theta=\frac{\pi}{2}$

$\int_0^{\frac{\pi}{2}}\sin x\ dx = 1$

17. Consider the line L given by the equation $\frac{x-3}{2}=\frac{y-1}{1}=\frac{z-2}{1}$. Let Q be the mirror image of the point (2,3, –1) with respect to L. Let a plane P be such that it passes through Q, and the line L is perpendicular to P. Then which of the following points is on the plane P ?

(1) (1, 1, 2)

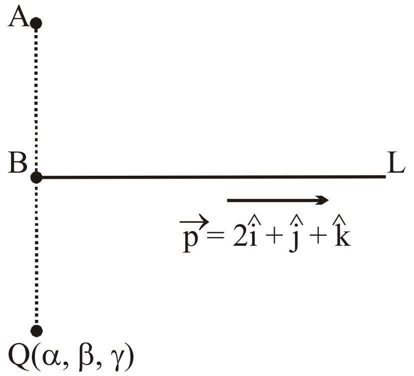
(2) (1, 2, 2)

(3) (1, 1, 2)

(4) (–1, 1, 2)

Ans. Official Answer NTA (2)

Sol.



$B(2\lambda+3,\ \lambda+1,\ \lambda+2)$

$\overrightarrow{AB} = (2\lambda+1)\hat{i} + (\lambda-2)\hat{j} + (\lambda+3)\hat{k}$

$\overrightarrow{AB} \cdot \vec{P} = 0$

$2(2\lambda+1) + \lambda - 2 + \lambda + 3 = 0$

$6\lambda = -3$

$\lambda = \frac{-1}{2}$

$B\left(2, \frac{1}{2}, \frac{3}{2}\right)$

$\frac{\alpha+2}{2} = 2,\quad \frac{\beta+3}{2} = \frac{1}{2},\quad \frac{\gamma-1}{2} = \frac{3}{2}$

$\alpha = 2,\quad \beta = 2,\quad \gamma = 2$

equation of plane 2x + y + z = of passing through (2, –2, 4)

d = 6

$2x + y + z = 6$

18. If the mean and variance of six observations 7, 10, 11, 15, a, b are 10 and $\frac{20}{3}$, respectively, then the value of |a – b| is equal to :

(1) 11

(2) 1

(3) 7

(4) 9

Ans. Official Answer NTA (2)

Sol. $\text{mean} = \frac{7+10+11+15+a+b}{6} = 10$

$4 + b = 17$

$\text{variance} = \frac{7^2+10^2+11^2+15^2+a^2+b^2}{6} - 100 = \frac{20}{3}$

$a^2 + b^2 = 145$

$(a+b)^2 - 2ab = 145$

$ab = 72$

$(a-b)^2 = (a+b)^2 - 4ab$

$|a-b| = 1$

19. For the natural numbers m, n, if $(1-y)^m (1+y)^n = 1 + a_1y + a_2y^2 + ... + a_{m+n} y^{m+n}$ and $a_1 = a_2 = 10$, then the value of (m + n) is equal to :

(1) 64

(2) 100

(3) 88

(4) 80

Ans. Official Answer NTA (4)

Sol. coeff. of $y = a_1 = {}^mC_0\, {}^nC_1 - {}^mC_1\, {}^nC_0 = 10$

$n - m = 10$ ....(1)

coeff. of $y = a_2 = {}^mC_0\, {}^nC_2 + {}^mC_2\, {}^nC_0 - {}^mC_1\, {}^nC_1 = 10$

$\frac{n(n-1)}{2} + \frac{m(m-1)}{2} - mn = 10$

$n^2 + m^2 - 2mn - n - m = 20$

$(m-n)^2 - n - m = 20$

$n + m = 80$

20. If $f : R \to R$ is given by $f(x) = x + 1$, then the value of

$\lim_{n\to\infty} \frac{1}{n}\left[f(0) + f\left(\frac{5}{n}\right) + f\left(\frac{10}{n}\right) + ... + f\left(\frac{5(n-1)}{n}\right)\right]$, is :

(1) $\frac{3}{2}$

(2) $\frac{1}{2}$

(3) $\frac{7}{2}$

(4) $\frac{5}{2}$

Ans. Official Answer NTA (3)

Sol. $\lim_{n\to\infty}\frac{1}{n}\left[f\left(\frac{0}{n}\right)+f\left(\frac{5}{n}\right)+f\left(\frac{10}{n}\right)+........+f\left(\frac{5(n-1)}{n}\right)\right]$

$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1}f\left(\frac{5r}{n}\right)$

$=\int_0^1 f(5x)\,dx=\int_0^1(5x+1)$

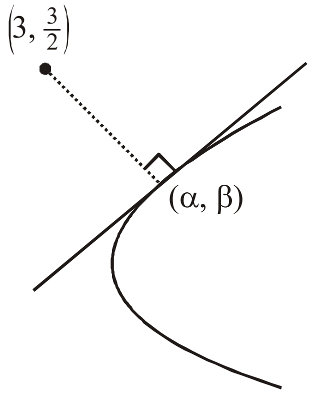
$=\left[5\frac{x^2}{2}+x\right]_0^1=\frac{7}{2}$

SECTION - B

1. If the point on the curve $y^2=6x$, nearest to the point $\left(3,\frac{3}{2}\right)$ is (α,β), then $2(\alpha+\beta)$ is equal to __________.

Ans. Official Answer NTA (9)

Sol.



$y^2 = 6n$
$\beta^2 = 6\alpha$

$y^2 = 6n$

$2y\frac{dy}{dx} = 6$

$\frac{dy}{dx} = \frac{3}{y} = \frac{3}{\beta}$

$\frac{\beta - \frac{3}{2}}{\alpha - 3} = \frac{-\beta}{3}$

$3\beta - \frac{9}{2} = -\alpha b + 3\beta$

$\alpha\beta = \frac{9}{2}$

$\beta^2 = 6\alpha$

$\beta = \frac{9}{2\alpha}$

$6\alpha = \frac{81}{4\alpha^2}$

$\alpha^3 = \frac{27}{8} \quad \Rightarrow \quad \alpha = \frac{3}{2}$

$\beta = \pm 3$

$\alpha = \frac{3}{2} \quad \beta = 3$

$2(\alpha + \beta) = 9$

2. For $k \in N$, let $\frac{1}{\alpha(\alpha+1)(\alpha+2).....(\alpha+20)} = \sum_{k=0}^{20} \frac{A_k}{\alpha+k}$, where $\alpha > 0$. Then the value of $100\left(\frac{A_{14}+A_{15}}{A_{13}}\right)^2$ is equal to_________.

Ans. Official Answer NTA (9)

Sol. by partial fraction

for A_{14} put $\alpha = -14$ in $\frac{1}{\alpha(\alpha+1)(\alpha+2)........(2+20)}$

$$A_{14} = \frac{1}{(-14)(-13)........(-1)(1)(2)........(6)}$$

$$A_{15} = \frac{1}{(-15)(-14)........(-1)(1)(2)........(5)}$$

$$A_{13} = \frac{1}{(-13)(-12)........(-1)(1)(2)........(7)}$$

$$100\left(\frac{A_{14}+A_{15}}{A_{13}}\right)^2 = 100\left\{\frac{\frac{-15+6}{(-15)(-14)........(-1)(1)(2)........(6)}}{\frac{1}{(-13)(-12)........(-1)(1)(2)........(7)}}\right\}^2$$

$$= 100\left(\frac{\frac{-9}{(-15)(-14)}}{\frac{1}{7}}\right)^2$$

$$= 100\left(\frac{-3}{10}\right)^2 = 9$$

3. The number of solutions of the equation $\log_{(x+1)}(2x^2+7x+5)+\log_{(2x+5)}(x+1)^2-4=0, x>0,$ is _______.

Ans. Official Answer NTA (1)

Sol. $\log_{(x+1)}(2x+5)(x+1)+2\log_{(2x+5)}(x+1)=4$

$1+\log_{(x+1)}(2x+5)+2\log_{(x+1)}(2x+5)=4$

put $\log_{(x+1)}(2x+5)=t$

$t+\frac{2}{t}=3$

$t^2-3t+2=0$

$t=1$ or $t=2$

$\log_{(x+1)}(2x+5)=1$ or $\log_{(x+1)}(2x+5)=2$

$2x+5=x+1$ or $2x+5=x^2+2x+1$

$x=-4$ $\quad$ $x^2=4$

Rejected

$x=2$ or $x=-2$

Rejected

4. If $\lim\limits_{x\to 0}\dfrac{axe^{x}-\beta\log_e(1+x)+\gamma x^2e^{-x}}{x\sin^2 x}=10,\ \alpha,\ \beta,\ \gamma\in R,$ then the value of $\alpha+\beta+\gamma$ is ___________.

Ans. Official Answer NTA (3)

Sol. $$\lim_{n\to 0}\frac{\alpha x\left(1+x+\frac{x^2}{2!}+.........\right)-\beta\left(x+\frac{x^2}{2}+\frac{x^3}{3}+.........\right)-\gamma x^2\left(1+x+\frac{x^2}{2!}-.........\right)}{x^3}$$

$\alpha-\beta=0 \quad \Rightarrow \alpha=\beta$

$\alpha+\frac{\beta}{2}+\gamma=0 \quad \Rightarrow \gamma=\frac{-3\beta}{2}$

$\frac{\alpha}{2}-\frac{\beta}{3}-\gamma=10$

$\beta-6,\ \alpha=6,\ \gamma=-9$

$\alpha+\beta+\gamma=3\alpha$

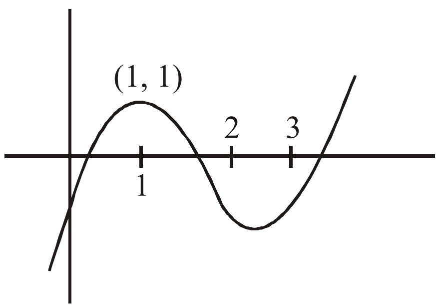
5. Let a function $g:[0,4]\to R$ be defined as $g(x)=\begin{cases}\max\{t^3-6t^2+9t-3\}, & 0\le x\le 3\\ 0\le t\le x & \\ 4-x & ,3<x\le 4\end{cases}$, then the number of points in the interval (0, 4) where g(x) is NOT differentiable, is ____________.

Ans. Official Answer NTA (1)

Sol. $y = t^3 - 6t^2 + 9t - 3$

$\frac{dy}{dx}=3t^2-12t+9=0$

$t = 1 \qquad t = 3$



$$g(x) = \begin{cases} x^3 - 6x^2 + 9x - 3 & 0 \le n \le 1 \\ 1 & 1 < x \le 3 \\ 4 - x & 3 < x \le 4 \end{cases}$$

$$g'(x) = \begin{cases} 3x^2 - 12x + 9 & 0 \le x \le 1 \\ 0 & 1 < n \le 3 \\ 4 & 3 < x \le 4 \end{cases}$$

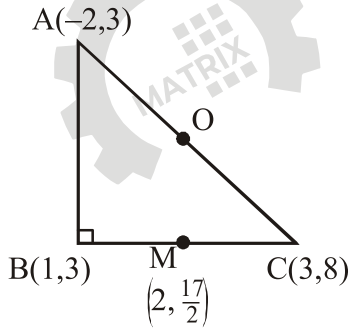
$g'(1^-) = g'(1^+)$ but $g'(3^-) \neq g'(3^+)$

nondifferentiable at one point

6. Consider a triangle having vertices A(–2, 3), B(1, 9) and C(3, 8). If a line L passing through the circum-center of triangle ABC, bisects line BC, and intersects y-axis at point $\left(0, \frac{\alpha}{2}\right)$, then the value of real number α is _______.

Ans. Official Answer NTA (9)

Sol.



$AC = 5\sqrt{2}$

$AB = \sqrt{45}$

$BC = \sqrt{5}$

Circumcentre is $\left(\frac{1}{2}, \frac{11}{2}\right)$

$$M_{OM} = \frac{\frac{17}{2} - \frac{11}{2}}{2 - \frac{1}{2}}$$

equation of OM

$$y - \frac{17}{2} = 2\,(x - 2)$$

$2y - 17 = 4x - 8$

$4y - 2y + 9 = 0$

put $\left(0, \frac{\alpha}{2}\right)$

$-\alpha + 9 = 0$

$\alpha = 9$

7. Let $A = \{a_{ij}\}$ be a 3×3 matrix, where $a_{ij} = \begin{cases} (-1)^{j-i} & \text{if } i < j, \\ 2 & \text{if } i = j, \\ (-1)^{i+j} & \text{if } i > j, \end{cases}$ then det(3 Adj(2 A^{-1})) is equal to _______.

Ans. Official Answer NTA (108)

Sol. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$|A| = 4$

$$\left|3\text{adj}\left(2A^{-1}\right)\right| = 3^3\left|\text{Adj}\left(2A^{-1}\right)\right|$$

$$= 3^3\left|2A^{-1}\right|^2$$

$$= 3^3 \times 2^6 \times \left|A^{-1}\right|^2$$

$$= 3^3 \times 2^6 \times \frac{1}{|A|^2} = 108$$

8. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_1 = 1, a_2 = 1$ and $a_{n+2} = 2a_{n+1} + a_n$ for all $n > 1$. Then the value of 47 $\sum_{n=1}^{\infty} \frac{a_n}{2^{3n}}$ is equal to _________.

Ans. Official Answer NTA (7)

Sol. $a_{n+2} = 2a_{n+1} + a_n$

$\frac{a_{n+2}}{8^n} = 2\frac{a_{n+1}}{8^n} + \frac{a_n}{8^n}$

$64\sum_{n=1}^{\infty}\left(\frac{a_{n+2}}{8^{n+2}}\right) = 16\sum_{n=1}^{\infty}\left(\frac{a_{n+1}}{8^{n+1}}\right) + \sum_{n=1}^{\infty}\frac{a_n}{8^n}$

Let $\sum_{n=1}^{\infty}\left(\frac{a_n}{8^n}\right) = p$

$64\left(p - \frac{a_1}{8} - \frac{a_2}{8^2}\right) = 16\left(p - \frac{a_1}{8}\right) + p$

$p = \frac{7}{47}$

$47\,p = 7$

9. For $p > 0$, a vector $\overrightarrow{v_2} = 2\hat{i} + (p+1)\hat{j}$ is obtained by rotating the vector $\overrightarrow{v_1} = \sqrt{3}\,p\,\hat{i} + \hat{j}$ by an angle θ about origin in counter clockwise direction. If $\tan\theta = \frac{(\alpha\sqrt{3} - 2)}{4\sqrt{3} + 3}$, then the value of α is equal to _______.

Ans. Official Answer NTA (6)

Sol. $\left|\overrightarrow{V}_1\right| = \left|\overrightarrow{V}_2\right|$

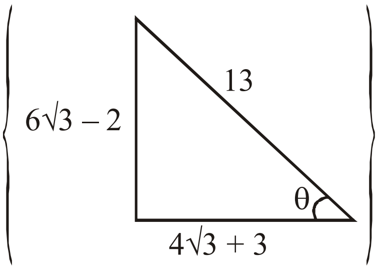
$3P^2 + 1 = 4 + (P + 1)^2$

$P = 2$ or $P = -1$ (rejected)

$\overrightarrow{V}_1 = 2\sqrt{3}\ \hat{i} + \hat{j}$

$\overrightarrow{V}_2 = 2\hat{i} + 3\hat{j}$

$\cos\theta = \frac{\overrightarrow{V}_1.\overrightarrow{V}_2}{\left|\overrightarrow{V}_1\right|\left|\overrightarrow{V}_2\right|} = \frac{4\sqrt{3} + 3}{13}$



$\tan\theta = \frac{6\sqrt{3} - 2}{4\sqrt{3} + 3}$

$\alpha = 6$

10. Let a curve $y = y(x)$ be given by the solution of the differential equation

$\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right)dx = \sqrt{e^{2x} - 1}\,dy$

If it intersects y-axis at $y = -1$, and the intersection point of the curve with x-axis is $(\alpha, 0)$, then e^{α} is equal to ________.

Ans. Official Answer NTA (2)

Sol. $\cos\left(\frac{1}{2}\cos^{-1}\left(e^{-x}\right)\right)dx = \sqrt{e^{2x}-1}\ dy$

Let $\cos^{-1} e^{-x} = \theta$

$\cos\theta = e^{-x}$

$\cos\frac{\theta}{2} = \sqrt{\frac{e^{-x}+1}{2}}$

$\sqrt{\frac{e^{-x}+1}{2}}\ dx = \sqrt{e^{x}+1}\ \sqrt{e^{x}-1}\ dy$

$\sqrt{\frac{1+e^{x}}{2e^{x}}}\ dx = \sqrt{e^{x}+1}\ \sqrt{e^{x}-1}\ dy$

$\int \frac{dx}{\sqrt{2e^{x}}\sqrt{e^{x}-1}} = \int dy$

$\frac{1}{\sqrt{2}}\int \frac{e^{-x}dx}{1-e^{-x}} = y+c$

$1-e^{-x} = t^2$

$e^{-x}\ dx = 2t\ dx$

$\frac{1}{\sqrt{2}}\int \frac{2t\ dt}{t} = y+c$

$\sqrt{2}\sqrt{1-e^{-x}} = y+c$

$\sqrt{2}\sqrt{1-e^{-x}} = y+1$

$y = 0$

$e^{x} = 2$

$c = 1$