JEE Main March 2021

Question Paper With Text Solution

18 March. | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN MARCH 2021 | 18TH MARCH SHIFT-2

SECTION – A

1. Let f: R–{3} $\rightarrow$ R – {1} be defined by $f(x)=\frac{x-2}{x-3}$ Let g : R $\rightarrow$ R be given as g(x) = 2x –3. Then the sum of all the values of x for which $f^{-1}(x)+g^{-1}(x)=\frac{13}{2}$ is equal to.

(1) 3 (2) 5 (3) 7 (4) 2

Ans. Official Answer NTA (2)

Sol. $y=\frac{x-2}{x-3}$ $\qquad$ $y = 2x - 3$

$xy - 3y = x - 2$ $\qquad$ $x=\frac{y+3}{2}$

$x(y-1) = 3y - 2$ $\qquad$ $g^{-1}(x)=\frac{x+3}{2}$

$x=\frac{3y-2}{y-1}$

$f^{-1}(x)=\frac{3x-2}{x-1}$

$f^{-1}(x)+g^{-1}(x)=\frac{13}{2}$

$\frac{3x-2}{x-1}+\frac{x+3}{2}=\frac{13}{2}$

$\frac{6x-4+(x+3)(x-1)}{(x-1)}=13$

$6x - 4 + x^2 + 2x - 3 = 13x - 13$

$x^2 + 8x - 7 - 13x + 13 = 0$

$x^2 - 5x + 6 = 0$

$(x - 2)(x - 3) = 0$

$x = 2, 3$

sum = 2 + 3 = 5

2. Define a relation R over a class of $n \times n$ real matrices A and B as "ARB iff there exists a non–singular matrix P such that $PAP^{-1} = B$" Then which of the following is true ?

(1) R is reflexive, transitive but not symmetric

(2) R is an equivalence relation

(3) R is symmetric, transitive but not reflexive,

(4) R is reflexive, symmetric but not transitive

Ans. Official Answer NTA (2)

Sol. For reflexive

$ARA \Rightarrow PAP^{-1} = A$ ...... (1) must be true

For P = I, eq. (1) is true so 'R' is reflexive.

For symmetric

$ARB \Leftrightarrow PAP^{-1} = B$ ...... (1) is true

For BRA iff $PBP^{-1} = A$ .... (2) must be true

$\because PAP^{-1} = B$

$P^{-1} PA P^{-1} = P^{-1}B$

$I A P^{-1}P = P^{-1}BP$

$A = P^{-1}BP$ .... (3)

From (2) and (3) $PBP^{-1} = P^{-1}BP$ can be true for some $P = P^{-1}$

$\Rightarrow P^2 = I$ (det (P) $\neq 0$)

So, R is symmetric

For transitive

$ARB \Leftrightarrow PAP^{-1} = B$ ..... is true

$BRC \Leftrightarrow PBP^{-1} = C$ ..... is true

Now $PPA P^{-1} P^{-1} = C$

$P^2A (P^2)^{-1} = C \Leftrightarrow ARC$

So R is transitive relation

$\Rightarrow$ Hence, R is equivalence

3. Let $f : R \rightarrow R$ be a function defined as $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin 2x}{2x} & ,\text{if} \quad x < 0 \\ b & ,\text{if} \quad x = 0 \\ \dfrac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} & ,\text{if} \quad x > 0 \end{cases}$. If f is continuous at $x = 0$, then the value of value of a + b is equal to:

(1) –3 (2) –2 (3) $-\frac{3}{2}$ (4) $-\frac{5}{2}$

Ans. Official Answer NTA (3)

Sol. $\lim_{x\to 0^-} \frac{\sin(a+1)x + \sin 2x}{2x}$

$\lim_{x\to 0^-} \frac{\sin(a+1)x}{2(a+1)x} \times (a+1) + \frac{\sin 2x}{2x}$

$= \frac{a+1}{2} + 1 = b$

a + 3 = 2b

$b = \frac{a+3}{2}$

$\lim_{x\to 0^+} \frac{\sqrt{x+bx^3} - \sqrt{x}}{bx^{5/2}} = \lim_{x\to 0^+} \frac{\sqrt{1+bx^2} - 1}{bx^2}$

$\lim_{x\to 0^+} \frac{\frac{bx^2}{2}}{bx^2} = \frac{1}{2}$

$b = \frac{1}{2}$ $\quad\quad \frac{a+3}{2} = \frac{1}{2}$

a = –2

$a + b = -2 + \frac{1}{2} = \frac{-3}{2}$

4. Let in a series of 2n observations, half of them are equal to a and remaining half are equal to – a. Also by adding a constant b in each of these observations, the mean and standard deviation of new set become 5 and 20 respectively. Then the value of $a^2 + b^2$ is equal to :

(1) 425 (2) 925 (3) 650 (4) 250

Ans. Official Answer NTA (1)

Sol. Let observations are denoted by xi

For $1 \le i < 2n$

$\bar{x} = \frac{\Sigma x_i}{2n} = \frac{(a+a+.......+a)-(a+a+.....+a)}{2n}$

$\bar{x} = 0$

and $(\sigma_x)^2 = \frac{\Sigma(x_i)^2}{2n} - (\bar{x})^2 = \frac{a^2 + a^2 + ... + a^2}{2n} - 0 = a^2$

$\Rightarrow \sigma_x = a$

Now, adding a constant 'b' then,

$\bar{y} = \bar{x} + b = 5$]

$\Rightarrow b = 5$

and $\sigma_y = \sigma_x$ (No change in S.D.)

$\Rightarrow a = 20$

$\Rightarrow a^2 + b^2 = 425$

5. Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors perpendicular to each other and $|\vec{a}| = |\vec{b}|$. If $|\vec{a} \times \vec{b}| = |\vec{a}|$, then the angle between the vectors $\left(\vec{a} + \vec{b} + (\vec{a} \times \vec{b})\right)$ and $\vec{a}$ is equal to :

(1) $\sin^{-1}\left(\frac{1}{\sqrt{6}}\right)$ (2) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (3) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (4) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Ans. Official Answer NTA (3)

Sol. $|\vec{a}| = |\vec{b}|$

$|\vec{a} \times \vec{b}| = |\vec{a}|$

$\vec{a} \perp \vec{b}$

$|\vec{a} \times \vec{b}| = |\vec{a}|$

$|\vec{a}||\vec{b}| \sin 90^\circ = |\vec{a}| \Rightarrow |\vec{b}| = 1$

$\vec{a}$ and $\vec{b}$ are two $\perp$er unit vectors

$\vec{a} = \hat{i}$, $\vec{b} = \hat{j}$

$\vec{a} \times \vec{b} = \hat{k}$

$\vec{c} = \left(\hat{i} + \hat{j} + \hat{k}\right)$, $\vec{a} = \hat{i}$

$\vec{a} \cdot \vec{c} = |\vec{a}||\vec{c}| \cos\theta$

$1 = 1 \times \sqrt{3} \cos\theta$

$\frac{1}{\sqrt{3}} = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

6. The area bounded by the curve $4y^2 = x^2(4-x)(x-2)$ is equal to :

(1) $\frac{3\pi}{8}$ (2) $\frac{3\pi}{2}$ (3) $\frac{\pi}{16}$ (4) $\frac{\pi}{8}$

Ans. Official Answer NTA (2)

Sol. $4y^2 = x^2(4-x)(x-2)$

$$y = \pm\frac{x}{2}\sqrt{(4-x)(x-2)}$$

$(4-x)(x-2) \geq 0$

$(x-2)(x-4) \leq 0$

$x \in [2, 4]$

$$y_1 = \frac{x}{2}\sqrt{(4-x)(x-2)}$$

$$y_2 = \frac{-x}{2}\sqrt{(4-x)(x-2)}$$

Required area = $\int_2^4 (y_1 - y_2)\,dx$

Required area = $\int_2^4 x\sqrt{(4-x)(x-2)}\,dx = I$

$$I = \int_2^4 (6-x)\sqrt{(x-2)(4-x)}\,dx$$

$$2I = \int_2^4 6\sqrt{(x-2)(4-x)}\,dx$$

$$I = 3\int_2^4 \sqrt{(x-2)(4-x)}\,dx$$

$$I = 3\int_2^4 \sqrt{1-(x-3)^2}\,dx$$

$$I = 3\times\frac{\pi}{2} = \frac{3\pi}{2}$$

7. Let S_1 be the sum of first 2n terms of an arithmetic progression. Let S_2 be the sum of first 4n terms of the same aritmetic progression. If $(S_2 – S_1)$ is 1000, then the sum of the first 6n terms of the arithmetic progression is equal to :

(1) 7000 (2) 5000 (3) 3000 (4) 1000

Ans. Official Answer NTA (3)

Sol. $S_1 = \frac{2n}{2}\left(2a + \left(2n-1\right)d\right) = n\left(2a + (2n-1)d\right)$

$S_2 = \frac{4n}{2}\left(2a + \left(4n-1\right)d\right) = 2n\left(2a + (4n-1)d\right)$

$S_2 - S_1 = 1000$

$2n\ (2a + (4n - 1)d) - n\ (2a + (2n - 1)d) = 1000$

$n\ [4a + (4n - 1)2d - 2a - (2n - 1)d] = 1000$

$n\ [2a + d\ (8n - 2 - 2n + 1)] = 1000$

$n\ [2a + d\ (6n - 1)] = 1000$

$\frac{6n}{2}\left(2a + \left(6n-1\right)d\right) = 3000$

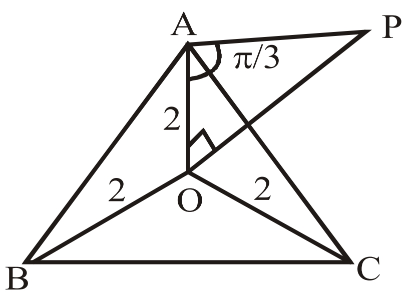
$S = 3000$

8. A pole stands vertically in side a triangular park ABC. Let the angle of elevation of the top of the pole from each corner of the park be $\frac{\pi}{3}$. If the radius of the circumcircle of ΔABC is 2, then the height of the pole is equal to :

(1) $\frac{1}{\sqrt{3}}$ (2) $2\sqrt{3}$ (3) $\frac{2\sqrt{3}}{3}$ (4) $\sqrt{3}$

Ans. Official Answer NTA (2)

Sol.



$\tan\frac{\pi}{3} = \frac{h}{2}$

$\sqrt{3} = \frac{h}{2} \Rightarrow h = 2\sqrt{3}$

9. Let y = y(x) be the solution of the differential equation $\frac{dy}{dx} = (y+1)\left((y+1)e^{x^2/2} - x\right)$, $0 < x < 2.1$, with $y(2) = 0$. Then the value of $\frac{dy}{dx}$ at x = 1 is equal to :

(1) $\frac{5e^{1/2}}{(e^2+1)^2}$ (2) $-\frac{2e^2}{(1+e^2)^2}$ (3) $\frac{-e^{3/2}}{(e^2+1)^2}$ (4) $\frac{e^{5/2}}{(1+e^2)^2}$

Ans. Official Answer NTA (3)

Sol. $\frac{dy}{dx} = (y+1)\left((y+1)e^{x^2/2} - x\right)$

$y + 1 = Y$

$\frac{dy}{dx} = \frac{dY}{dx}$

$\frac{dY}{dx} = Y\left(Ye^{x^2/2} - x\right)$

$\frac{dY}{dx} = Y^2e^{x^2/2} - xY$

$\frac{1}{Y}\frac{dY}{dx} = Ye^{x^2/2} - x$

$\frac{1}{Y} = P$

$\frac{dP}{dx} = \frac{-1}{Y^2}\frac{dY}{dx}$

$\frac{1}{Y^2}\frac{dY}{dx} = e^{x^2/2} - \frac{x}{Y}$

$\frac{-dP}{dx} = e^{x^2/2} - Px$

$\frac{dP}{dx} = -e^{x^2/2} + Px$

$\frac{dP}{dx} - Px = e^{x^2/2}$

I.F. $= e^{x^2/2}$

$-P = (x + c)\, e^{x^2/2}$

$\frac{-1}{Y} = (x + c)\, e^{x^2/2}$

$\frac{-1}{y+1} = (x + c)\, e^{x^2/2}$

$y + 1 = \frac{-1}{(x+c)\, e^{x^2/2}}$

$x = 2,\ y = 0$

$0 + 1 = \frac{-1}{(2+c)\, e^{x^2/2}}$

$(2 + c)\, e^{x^2/2} = -1$

$2 + c = \frac{-1}{e^{x^2/2}}$

$c = -2 \frac{-1}{e^{x^2/2}}$

$y + 1 = \frac{-1}{\left(x - 2 - \frac{1}{e^{x^2/2}}\right) e^{x^2/2}}$

$\left(\frac{dy}{dx}\right)_{\text{at } x=1} = \frac{-e^{3/2}}{\left(1+e^2\right)^2}$

10. Let a tangent be drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$ where $\theta \varepsilon \left(0, \frac{\pi}{2}\right)$. Then the value of θ such that the sum of intercepts on axes made by this tangent is minimum is equal to -

(1) $\frac{\pi}{3}$ (2) $\frac{\pi}{6}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{8}$

Ans. Official Answer NTA (2)

Sol. $\frac{x^2}{27} + y^2 = 1$

$\frac{2x}{27} + 2y\, y' = 0$

$\frac{x}{27} + y\, y' = 0 \Rightarrow yy' = \frac{-x}{27}$

$$y' = \frac{-x}{27y}$$

$$y'_{(3\sqrt{3}\cos\theta, \sin\theta)} = \frac{-3\sqrt{3}\cos\theta}{27\sin\theta} = \frac{-\sqrt{3}\cos\theta}{9\sin\theta}$$

$$y' = \frac{-\cot\theta}{3\sqrt{3}}$$

$$y - \sin\theta = \frac{-\cot\theta}{3\sqrt{3}}\left(x - 3\sqrt{3}\cos\theta\right)$$

Put $x = 0$

$$y - \sin\theta = \cos\theta \times \cot\theta$$

$$y = \cos\theta\cot\theta + \sin\theta = \frac{1}{\sin\theta}$$

Put $y = 0$

$$-\sin\theta = \frac{-\cot\theta}{3\sqrt{3}}\left(x - 3\sqrt{3}\cos\theta\right)$$

$$\sin\theta = \frac{\cos\theta}{3\sqrt{3}\sin\theta}\left(x - 3\sqrt{3}\cos\theta\right)$$

$$\frac{3\sqrt{3}\sin^2\theta}{\cos\theta} = x - 3\sqrt{3}\cos\theta$$

$$\frac{3\sqrt{3}\sin^2\theta + 3\sqrt{3}\cos^2\theta}{\cos\theta} = x$$

$$x = \frac{3\sqrt{3}}{\cos\theta}$$

$$x_{int} + y_{int} = \frac{1}{\sin\theta} + \frac{3\sqrt{3}}{\cos\theta}$$

$$= \cos ec\theta + 3\sqrt{3}\sec\theta$$

$$f(\theta) = 3\sqrt{3}\sec\theta + \operatorname{cosec}\theta$$

$$f'(\theta) = 3\sqrt{3} \times \sec\theta\tan\theta - \operatorname{cosec}\theta\cot\theta$$

$$= \frac{3\sqrt{3}}{\cos\theta} \times \frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\sin^2\theta}$$

$$3\sqrt{3} = \cot^3\theta$$

$$3^{3/2} = \cot^3\theta$$

$$\cot\theta = \sqrt{3}$$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \frac{\pi}{6}$

11. Let the system of linear equations

$4x + \lambda y + 2z = 0$

$2x - y + z = 0$

$\mu x + 2y + 3z = 0$, $\lambda, \mu \in R$.

has a non-trivial solution. Then which of the following is true ?

(1) $\lambda = 3, \mu \in R$ (2) $\mu = -6, \lambda \in R$ (3) $\mu = 6, \lambda \in R$ (4) $\lambda = 2, \mu \in R$

Ans. Official Answer NTA (3)

Sol. $4x + \lambda y + 2z = 0$
$2x - y + z = 0$
$\mu x + 2y + 3z = 0 \quad \lambda, \mu \in R$

$$\begin{vmatrix} 4 & \lambda & 2 \\ 2 & -1 & 1 \\ \mu & 2 & 3 \end{vmatrix} = 0$$

$4(-3-2) - \lambda(6-\mu) + 2(4+\mu) = 0$
$-20 - 6\lambda + \lambda\mu + 8 + 2\mu = 0$
$\mu(\lambda + 2) - 6\lambda = 12$
$6(\lambda + 2) - 6\lambda = 12$
$12 = 12 \Rightarrow \mu = 6, \lambda \in R$

12. Let in a Binomial distribution, consisting of 5 independent trials, probabilites of exactly 1 and 2 successes be 0.4096 and 0.2048 respectively. Then the probability of getting exactly 3 successes is equal to:

(1) $\frac{80}{243}$ (2 $\frac{128}{625}$ (3) $\frac{32}{625}$ (4) $\frac{40}{243}$

Ans. Official Answer NTA (3)

Sol. $P(X = 1) = 5_{C_1} \times p \times q^4 = 0.4096$

$P(X = 2) = 5_{C_2} \times p^2 \times q^3 = 0.2048$

$\Rightarrow \frac{q}{2p} = 2$

$\Rightarrow q = 4p$ and $p + q = 1$

$p = \frac{1}{5},\ q = \frac{4}{5}$

$P(X = 3) = {}^{5}C_{3} \times \left(\frac{1}{5}\right)^{3} \times \left(\frac{4}{5}\right)^{2} = \frac{32}{625}$

13. If $15\sin^4\alpha + 10\cos^4\alpha = 6$, for some $\alpha \in R$, then the value of $27\sec^6\alpha + 8\operatorname{cosec}^6\alpha$ is equal to :

(1) 400 (2) 350 (3) 250 (4) 500

Ans. Official Answer NTA (3)

Sol. $15\sin^4\alpha + 10\cos^4\alpha = 6$

$15\tan^4\alpha + 10 = 6\sec^4\alpha$

$15\tan^4\alpha + 10 = 6(1 + \tan^2\alpha)^2$

$15\tan^4\alpha + 10 = 6 + 6\tan^4\alpha + 12\tan^2\alpha$

$9\tan^4\alpha - 12\tan^2\alpha + 9 = 0$

$\tan^2\alpha = t$

$(3t - 2)^2 = 0 \Rightarrow t = \frac{2}{3}$

$\tan^2\alpha = \frac{2}{3} \Rightarrow \tan\alpha = \pm\frac{\sqrt{2}}{\sqrt{3}}$

$1 + \tan^2\alpha = 1 + \frac{2}{3} = \frac{5}{3}$

$\sec^2\alpha = \frac{5}{3}$

$H = \sqrt{5}$

$\sin\alpha = \pm\frac{\sqrt{2}}{\sqrt{5}}$

$\sin^2\alpha = \frac{2}{5}$

$\operatorname{cosec}^2\alpha = \frac{5}{2}$

$27 \times \left(\frac{5}{3}\right)^{3} + 8\left(\frac{5}{2}\right)^{3} = 125 + 125 = 250$

14. Let $g(x)=\int_0^x f(t)dt$, where f is continuous function in [0, 3] such that $\frac{1}{3}\le f(t)\le 1$ for all t ∈ [0, 1] and $0\le f(t)\le \frac{1}{2}$ for all t ∈ (0, 3]. The largest possible interval in which g(3) lies is :

(1) $\left[-1,-\frac{1}{2}\right]$ (2) [1, 3] (3) $\left[\frac{1}{3},2\right]$ (4) $\left[-\frac{3}{2},-1\right]$

Ans. Official Answer NTA (3)

Sol. $\frac{1}{3}\le f(x)\le 1\,\forall\, t\in[0,1]$

$0\le f(t)\le \frac{1}{2}\ \forall\ t\in(1,3]$

$g(3)=\int_0^3 f(t)dt=\int_0^1 f(t)dt+\int_1^3 f(t)\,dt$

$\int_0^1 f(t)\,dt \Rightarrow \int_0^1 \frac{1}{3}dt \le \int_0^1 f(t)dt \le \int_0^1 1dt$

$\Rightarrow \frac{1}{3}\le \int_0^1 f(t)dt \le 1$ ..... (i)

$\int_1^3 f(t)dt \Rightarrow \int_1^3 0\,dt \le \int_1^3 f(t)dt \le \int_1^3 \frac{1}{2}dt$

$0\le \int_1^3 f(t)\,dt \le 1$ .... (ii)

(i) + (ii)

$\frac{1}{3}\le \int_0^1 f(t)\,dt+\int_1^3 f(t)\,dt \le 2$

$\frac{1}{3}\le g(3)\le 3$

15. Let $S_1 : x^2 + y^2 = 9$ and $s_2 : (x-2)^2 + y^2 = 1$. Then the locus of center of a variable circle S which touches S_1 internally and S_2 externally always passes through the points :

(1) $\left(2, \pm\frac{3}{2}\right)$ (2) $\left(0, \pm\sqrt{3}\right)$ (3) $\left(\frac{1}{2}, \pm\frac{\sqrt{5}}{2}\right)$ (4) $\left(1, \pm 2\right)$

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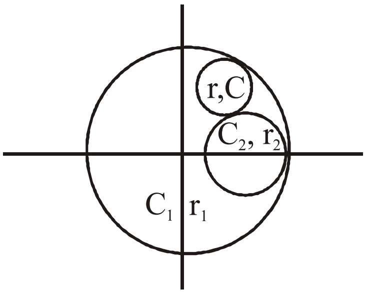
Ans. Official Answer NTA (1)

Sol.



$CC_1 = r_1 - r$

$CC_1 = r_2 + r$

$CC_1 + CC_2 = r_1 + r_2$

$CC_1 + CC_2 =$ constant

Locus is ellipse

$2a = 4 \Rightarrow a = 2$

$2ae = 2$

$e = 1/2$

$e^2 = 1 - \frac{b^2}{a^2}$

$\frac{1}{4} = 1 - \frac{b^2}{4}$

$\frac{b^2}{4} = \frac{3}{4} \Rightarrow b^2 = 3$

$\frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$

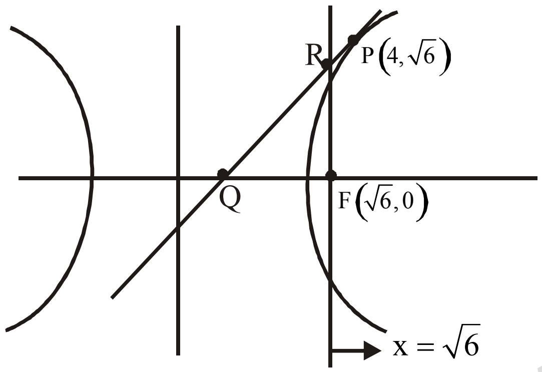
Option 1 satisfies it.

16. Consider a hyperbola H : $x^2 - 2y^2 = 4$ Let the tangent at a point $P(4,\sqrt{6})$ meet the x-axis at Q and latus rectum at $R(x_1,y_1)$, $x_1>0$. If F is focus of H which is nearer to the point P, then the area of ΔQFR is equal to.

(1) $4\sqrt{6}$ (2) $\sqrt{6}-1$ (3) $\frac{7}{\sqrt{6}}-2$ (4) $4\sqrt{6}-1$

Ans. Official Answer NTA (3)

Sol.



H : $x^2 - 2y^2 = 4$

$$\frac{x^2}{4}-\frac{y^2}{2}=1$$

$$e=\sqrt{1+\frac{b^2}{a^2}}=\frac{\sqrt{3}}{\sqrt{2}}$$

$F = (ae, 0) = \left(\sqrt{6},0\right)$

Tangent at P $\Rightarrow$ $2x - y\sqrt{6} = 2$

Q = (1,0)

At R $\Rightarrow$ latus rectum meets the tangent

$$R=\left(\sqrt{6},\frac{2}{\sqrt{6}}\left(\sqrt{6}-1\right)\right)$$

$$\text{Area}=\frac{1}{2}\left(\sqrt{6}-1\right)\left(\frac{2}{\sqrt{6}}\left(\sqrt{6}-1\right)\right)=\frac{7}{\sqrt{6}}-2$$

17. Let a complex number be $w = 1-\sqrt{3}i$. Let another complex number z be such that $|zw| = 1$ and arg(z)– arg(w) $= \frac{\pi}{2}$. Then the area of the triangle with vertices origin, z and w is equal to:

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) 2 (4) 4

Ans. Official Answer NTA (1)

Sol. $w = 1 - \sqrt{3}\,i \qquad |z\,w| = 1$

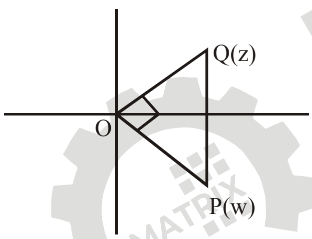
$\arg(z) - \arg(w) = \frac{\pi}{2}$

$|w| = 2$

$|z|\ |w| = 1 \Rightarrow |z| = \frac{1}{2}$

$\arg(z) = \frac{\pi}{2} + \arg|w|$

area $= \frac{1}{2} \times (OP)(OQ) = \frac{1}{2} \times 2 \times \frac{1}{2} = \frac{1}{2}$

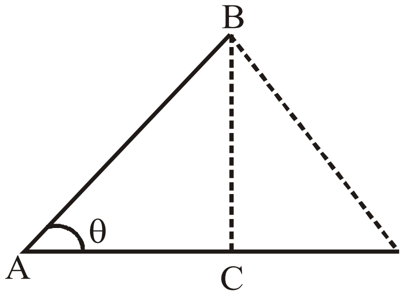


18. In a triangle ABC. If $|\overrightarrow{BC}| = 8, |\overrightarrow{CA}| = 7, |\overrightarrow{AB}| = 10$, then the projection of the vector $\overrightarrow{AB}$ on $\overrightarrow{AC}$ is equal to:

(1) $\frac{127}{20}$ (2) $\frac{115}{16}$ (3) $\frac{25}{4}$ (4) $\frac{85}{14}$

Ans. Official Answer NTA (4)

Sol.



$\left|\overrightarrow{BC}\right| = 8 \quad \left|\overrightarrow{CA}\right| = 7 \quad \left|\overrightarrow{AB}\right| = 10$

Projection = $\left|\overrightarrow{AB}\right| \cos\theta$

Projection = $10\cos\theta$

$$\cos = \frac{\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{CA}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\times\left|\overrightarrow{AB}\right|\left|\overrightarrow{CA}\right|}$$

$$= \frac{100+49-64}{2\times10\times7} = \frac{85}{10\times2\times7}$$

Projection $= 10\times\frac{85}{10\times2\times7} = \frac{85}{14}$

19. Let the centroid of an equilateral triangle ABC be at the origin. Let one of the sides of the equilateral triangle be along the straight line x + y = 3. If R and r be the radius of circumcircle and incircle respectively of ΔABC, then (R + r) is equal to :

(1) $3\sqrt{2}$ (2) $\frac{9}{\sqrt{2}}$ (3) $2\sqrt{2}$ (4) $7\sqrt{2}$

Ans. Official Answer NTA (2)

Sol. $r = \left|\frac{-3}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$

$r = 4R\sin A/2 \sin B/2 \sin C/2$

$\frac{3}{\sqrt{2}} = 4R\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$

$R = 3\sqrt{2}$

$R + r = \frac{3}{\sqrt{2}} + 3\sqrt{2} = \frac{9}{\sqrt{2}}$

20. If P and Q are two statements, then which of the following compound statement is a tautology ?

(1) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow Q$ (2) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow (P \wedge Q)$

(3) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$ (4) $((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$

Ans. Official Answer NTA (3)

Sol.

P	Q	$P \vee Q$	$P \vee Q$	$\sim P$	$(P \vee Q) \vee \sim P$
T	T	T	T	F	T
T	F	T	F	F	T
F	T	T	F	T	T
F	F	F	F	T	T

1. Let ${}^nC_r$ denote the binomial coefficient of x^r in the expansion of $(1 + x)^n$

If $\sum_{k=0}^{10}(2^2 + 3k)\ {}^{10}C_k = \alpha 2^{10} + \beta.2^{10}, \alpha, \beta \in R$, then $\alpha + \beta$ is equal to _____ .

Ans. Official Answer NTA (19)

Sol. $\sum_{k=0}^{10} 4 \times {}^{10}C_k + 3k \times {}^{10}C_k$

$= 4\sum_{k=0}^{10} {}^{10}C_k + 3\left(\sum_{k=0}^{10} k \times \frac{10}{k} \times {}^9C_{k-1}\right)$

$= 4 \times 2^{10} + 30\sum_{k=1}^{10} {}^9C_{k-1}$

$= 2^{12} + 30\left(2^9\right)$

$= 2^{12} + 30\left(2^9\right)$

$= 2^{12} + 15 \times 2^{10}$

$= \alpha \times 2^{10} + \beta \times 2^{10}$

$\beta = 15$

$\alpha = 4$

$\alpha + \beta = 19$

2. Let y = y(x) be the solution of the differential equation $xdy - ydx = \sqrt{(x^2 - y^2)}dx, x \geq 1$ with y(1) = 0. If the area bounded by the line x = 1, x = e^{π}, y = 0 and y = y(x) is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to______ .

Ans. Official Answer NTA (4)

Sol. $xdy - ydx = \sqrt{x^2 - y^2}\ dx$

$$\frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1 - \frac{y^2}{x^2}}\ dx$$

$$\int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 - \frac{y^2}{x^2}}} = \int \frac{dx}{x}$$

$$\sin^{-1}\left(\frac{y}{x}\right) = \ln|x| + c$$

$x = 1, y = 0 \Rightarrow c = 0$

$$\frac{y}{x} = \sin(\ln x)$$

$y = x \sin(\ln x)$

$$A = \int_1^{e^{\pi}} x \sin(\ln x)\ dx$$

$x = e^t$

$dx = e^t\ dt$

$$A = \int_0^{\pi} e^t \sin(\ln e^t) \times e^t dt$$

$$A = \int_0^{\pi} e^{2t} \sin t\ dt$$

$$A = \frac{e^{2t}}{5}(2\sin t - \cos t)\Big|_0^{\pi}$$

$$= \frac{e^{2\pi}}{5}(2\sin \pi - \cos \pi) - \frac{1}{5}(2\sin 0 - \cos 0)$$

$$A = \frac{e^{2\pi}}{5}(0 + 1) - \frac{1}{5}(-1)$$

$A = \frac{e^{2\pi}}{5} + \frac{1}{5} = \frac{1}{5}\left(e^{2\pi} + 1\right) = \alpha e^{2\pi} + \beta$

$\alpha = \frac{1}{5} \quad \beta = \frac{1}{5}$

$10\,(\alpha + \beta) = 10\left(\frac{1}{5} + \frac{1}{5}\right) = 4$

3. Let p(x) be a real polynomial of degree 3 which vanishes at x = –3, Let p(x) have local minima at x = 1, local maxima at x = –1 and $\int_{-1}^{1} P(x)dx = 18$, then the sum of all the coefficients of the polynomial P(x) is equal to ____ .

Ans. Official Answer NTA (8)

Sol. $P'(x) = a\,(x - 1)\,(x + 1) = a(x^2 - 1)$

$\frac{dP(x)}{dx} = a(x^2 - 1)$

$\int dP(x) = \int a\left(x^2 - 1\right)dx$

$P(x) = \frac{ax^3}{3} - ax + C$

$P(x) = a\left(\frac{x^3}{3} - x\right) + C$

$P(-3) = 0$

$0 = a\left(\frac{-27}{3} + 3\right) + C$

$0 = a\,(-6) + C$

$C = 6a$

$P(x) = a\left(\frac{x^3}{3} - x\right) + 6a$

$\int_{-1}^{1}\left(a\left(\frac{x^3}{3} - x\right) + 6a\right)dx = 18$

$\int_{-1}^{1}\frac{a}{3} \times x^3 dx - \int_{-1}^{1} ax\ dx + 60\int_{-1}^{1} dx\ = 18$

$$\frac{a}{12}(0)-\frac{a}{2}(1-1)+6a(1+1)=18$$

12a = 18

a = 3/2

$$P(x)=\frac{3}{2}\left(\frac{x^3}{3}-x\right)+9$$

$$=\frac{x^3}{2}-\frac{3x}{2}+9$$

Sum of coefficients $\frac{1}{2}-\frac{3}{2}+9=8$

4. Let P be a plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+5}{2}$ and parallel to the line $\frac{x-3}{4}=\frac{y-2}{-3}=\frac{z+5}{7}$ If the point (1,–1, α) lies on the plane P, then the value of $|5\alpha|$ is equal to _________.

Ans. Official Answer NTA (38)

Sol. $\vec{r}=(1,-6,-5)+\lambda(3,4,2)$

$$\vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 4 & 2\\ 4 & -3 & 7\end{vmatrix}=\hat{i}(28+6)-\hat{j}(21-8)+\hat{k}(-9-16)$$

$= 34\hat{i}-13\hat{j}-25\hat{k}$

$\vec{r}\cdot\vec{n}=\vec{a}\cdot\vec{n}$

$\vec{r}\cdot\left(34\hat{i}-13\hat{j}-25\hat{k}\right)=\left(\hat{i}-6\hat{j}-5\hat{k}\right).\left(34\hat{i}-13\hat{j}-25\hat{k}\right)$

34x – 13y – 25z = 34 + 78 + 125

34x – 13y – 25z = 237

34 + 13 – 25 α = 237

$$\alpha=\frac{-190}{25}$$

$$|5\alpha|=\left|5\times\frac{-190}{25}\right|=\frac{190}{5}=38$$

5. Let f : R→ R satisfy the equation $f(x + y) = f(x). f(y)$ for all x,y ∈ R and $f(x) \neq 0$ for any x ∈ R. If the function f is differentiable at x = 0 and $f'(0) = 3$, then $\lim_{h\to 0}\frac{1}{h}(f(h)-1)$ is equal to _______ .

Ans. Official Answer NTA (3)

Sol. $f(x + y) = f(x) f(y)$

$f(x) = a^x$

$f'(x) = a^x \ln a$

$f'(o) = a^0 \times \ln a = 3$

$\Rightarrow a = e^3$

$f(x) = e^{3x}$

$$\lim_{h\to 0}\frac{f(h)-1}{h} = \lim_{h\to 0}\left(\frac{e^{3h}-1}{3h}\right)\times 3 = 3$$

6. If $f(x)$ and g(x) are two polynomials such that the polynomial $P(x) = f(x^3) + x\, g(x^3)$ is divisible by $x^2 + x + 1$, then P(1) is equal to _____ .

Ans. Official Answer NTA (0)

Sol. $P(x) = f(x^3) + x\, g(x^3)$

$P(1) = f(1) + g(1)$ ....(1)

$P(x) = Q(x)\,(x^2 + x + 1)$

$P(\omega) = 0 = P(\omega^2)$

ω and ω^2 are non - real

cube roots of unity

$P(\omega) = f(\omega^3) + \omega\, g(\omega^3)$

$f(1) + \omega g(1) = 0$ ....(2)

$P(\omega^2) = f(\omega^6) + \omega^2\, g(\omega^6) = 0$

$f(1) + \omega^2\, g(1) = 0$ ....(3)

(3) + (2)

$\Rightarrow 2f(1) + (\omega + \omega^2)g(1) = 0$

$2f(1) = g(1)$ ....(4)

(2) – (3)

$\Rightarrow (\omega - \omega^2)\, g(1) = 0$

$g(1) = 0 = f(1)$ from - ....(4)

from (1) $\Rightarrow P(1) = f(1) + g(1) = 0$

7 The term independent of x in the expansion of $\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right]^{10}, x \neq 1$, is equal to ________.

Ans. Official Answer NTA (210)

Sol. $\left[\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right]^{10}$

$$\left(\frac{\left(x^{1/3}+1\right)\left(x^{2/3}-x^{1/3}+1\right)}{\left(x^{2/3}-x^{1/3}+1\right)}-\frac{\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}{\sqrt{x}\left(\sqrt{x}-1\right)}\right)^{10}$$

$$\left(x^{1/3}+1-1-\frac{1}{\sqrt{x}}\right)^{10}=\left(x^{1/3}-\frac{1}{x^{1/2}}\right)^{10}$$

$$T_{r+1}=10_{C_r}\left(x^{1/3}\right)^{10-v}\left(\frac{-1}{x^{1/2}}\right)^{r}$$

$$=10_{C_r}\, x\frac{10-r}{3}(-1)^r \times x^{-r/2}$$

$$\frac{10-r}{3}=\frac{r}{2}$$

$20 - 2r = 3r$

$r = 4$

$$T_5 = 10_{C_4}(-1)^4 = 10_{C_4} = \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} = 210$$

8. Let I be an identity matrix of order 2 × 2 and $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$. Then the value of $n \in N$ for which $P^n = 5I - 8P$ is equal to _____.

Ans. Official Answer NTA (6)

Sol. $P = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

$P^2 = \begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

$= \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}$

$P^3 = \begin{bmatrix} -1 & 1 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 5 & -3 \end{bmatrix}$

$= \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix}$

$5I - 8P = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 16 & -8 \\ 40 & -24 \end{bmatrix}$

$= \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$

$P^6 = \begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 10 & -7 \end{bmatrix}$

$= \begin{bmatrix} -11 & 8 \\ -40 & 29 \end{bmatrix}$

n = 6

9. If $\sum_{r=1}^{10} r!\left(r^3 + 6r^2 + 2r + 5\right) = \alpha(11!)$, , then the value of α is equal to _______.

Ans. Official Answer NTA (160)

Sol. $\sum_{r=1}^{10} \underline{|r}\left(r^3 + 6r^2 + 2r + 5\right) = \alpha\underline{|11}$

$\sum_{r=1}^{10} \underline{|r}\left((r+1)(r+2)(r+3) - 9(r+1) + 8\right)$

$$\sum_{r=1}^{10} \lfloor\underline{r+3} - 9\lfloor\underline{r+1} + 8\lfloor\underline{r}$$

$$\sum_{r=1}^{10} \lfloor\underline{r+3} - \lfloor\underline{r+1} - 8\lfloor\underline{r+1} + 8\lfloor\underline{r}$$

$$\sum_{r=1}^{10} \lfloor\underline{r+3} - \lfloor\underline{r+1} - 8\sum_{r=1}^{10} \lfloor\underline{r+1} - \lfloor\underline{r}$$

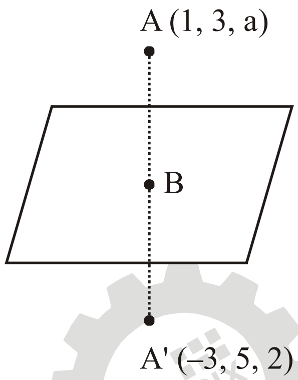
$$\Rightarrow \lfloor\underline{11}\left(12+156-8\right) = 160\lfloor\underline{11} = \alpha\lfloor\underline{11}$$

$\alpha = 160$

10. Let the mirror image of the point (1, 3, a) with respect to the plane $\vec{r}.\left(2\hat{i}-\hat{j}+\hat{k}\right)-b=0$ be (–3, 5, 2). Then, the value of |a + b| is equal to ______.

Ans. Official Answer NTA (1)

Sol.



$2x - y + z = b$

B is mid - point of AA' $= \left(-1, 4, \dfrac{a+2}{2}\right)$

$$-2-4+\frac{a+2}{2} = b$$

$-4-8+a+2 = 2b$

$a = 2b+10$

$A'A = 4\hat{i} - 2\hat{j} + (a-2)\hat{k}$

$$\frac{4}{2} = \frac{-2}{-1} = \frac{a-2}{1}$$

$a - 2 = 2 \Rightarrow a = 4$

$b = -3$

$|a+b| = |4-3| = 1$