JEE Main March 2021

Question Paper With Text Solution

17 March. | Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911
Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE MAIN MARCH 2021 | 17TH MARCH SHIFT-2

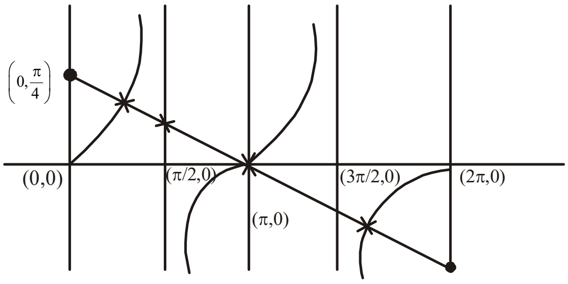
SECTION – A

1. The number of solutions of the equation $x + 2\tan x = \frac{\pi}{2}$ in the inteval $[0, 2\pi]$ is :

(1) 4 (2) 5 (3) 2 (4) 3

Ans. Official Answer NTA (4)

Sol.



$x + 2\tan x = \frac{\pi}{2}$

$2\tan x = \frac{\pi}{2} - x$

$\tan x = \frac{\pi}{4} - \frac{x}{2}$

If y = tan x

$y = \frac{\pi}{4} - \frac{x}{2}$

Clearly both graphs are intersecting at three distinct points, hence number of solutions are 3.

2. Let S_1, S_2 and S_3 be three sets defined as

$S_1 = \{z \in C : |z-1| \le \sqrt{2}\}$

$S_2 : \{z \in C : (Re(1-i)z) \ge 1\}$

$S_3 : \{z \in C : Im(z) \le 1\}$

Then the set $S_1 \cap S_2 \cap S_3$:

(1) has infinitely many elements (2) has exactly two elements

(3) has exactly three elements (4) is a singleton

Ans. Official Answer NTA (1)

Sol.

For S_1

$|z-1| \le \sqrt{2}$

$|x+iy-1| \le \sqrt{2}$

$(x-1)^2 + y^2 \le 2$

For S_2

$\text{Re}(1-i)Z \ge 1$

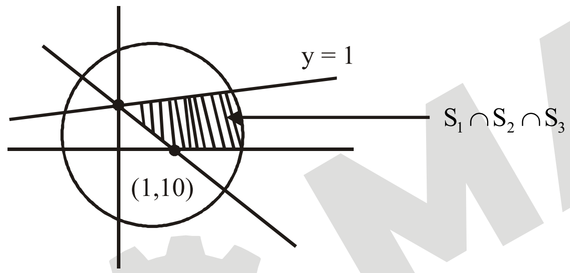
$\text{Re}((x+iy)(1-i)) \ge 1$

$x + y \ge 1$

For S_3

$\text{Im}(z) \le 1$

$y \le 1$



Hence no. of solution are infinite

3. If the integral $\int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$, where α, β, γ are integers and [x] denotes the greatest integer less than or equal to x, then the value of $\alpha + \beta + \gamma$ is equal :

(1) 0 (2) 20 (3) 25 (4) 10

Ans. Official Answer NTA (1)

Sol. $I = \int_0^{10} \frac{[\sin 2\pi x]}{e^{x-[x]}} dx$

$I = \int_0^{10\times 1} [\sin 2\pi\, x] e^{-\{x\}}\, dx$

$I = 10\int_0^1 [\sin\ 2\pi\, x] e^{-\{x\}} dx$

$I = 10\int_0^1 e^{-x} [\sin 2\pi x] dx$

$I = 10\int_0^{\frac{1}{2}} e^{-x} \times 0 \; dx + 10\int_{\frac{1}{2}}^{1} e^{-x} \; (-1) \; dx$

$I = -10\left(\frac{e^{-x}}{-1}\right)\Bigg]_{\frac{1}{2}}^{1}$

$I = 10\left(e^{-1} - e^{-\frac{1}{2}}\right) = \alpha e^{-1} + \beta e^{-\frac{1}{2}} + \gamma$

$\alpha = 10$
$\beta = -10$
$\gamma = 0$
$\alpha + \beta + \gamma = 0$

4. Let a computer program generate only the digit 0 and 1 to form a string of binary numbers with probability of occurence of 0 at even places be $\frac{1}{2}$ and probability of occurrence of 0 at the odd place be $\frac{1}{3}$.

Then the probability that '10' is followed '01' is equal to:

(1) $\frac{1}{9}$ (2) $\frac{1}{6}$ (3) $\frac{1}{3}$ (4) $\frac{1}{18}$

Ans. Official Answer NTA (1)

Sol. O E O E

0 1 1 0

or

E O E O

0 1 1 0

$p = \frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{3}$

$p = \frac{1}{9}$

5. The value of $\lim_{n\to\infty} \frac{[r]+[2r]+......+[nr]}{n^2}$, where r is non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to:

(1) 2r (2) 0 (3) r (4) $\frac{r}{2}$

Ans. Official Answer NTA (4)

Sol. By sandwich concept

$x - 1 < [x] \le x$

$1.\, r - 1 < [1.\, r] \le 1.\, r$

$2.\, r - 1 < [2.\, r] \le 2.\, r$

$n.\, r - 1 < [n.\, r] \le n.\, r$

By summation

$r\frac{n(n+1)n}{2} - n < [r] + [2r] + ...[nr] \le r\,(1 + 2 + ...+ n)|$

$\frac{\frac{r(n+1)n}{2} - n}{n^2} < [r] + [2r] + ...+ [nr] \le \frac{r\,n(n+1)}{2}$ division by n^2

$\lim_{n\to\infty}\frac{-n}{n^2} < \lim_{n\to\infty}\frac{[r]+[2r]+....[nr]}{n^2} \le \lim_{n\to\infty}\frac{r\;n(n+1)}{2n^2}$

$\frac{r}{2} < \frac{[r]+[2r]+....[nr]}{n^2} \le \frac{r}{2}$

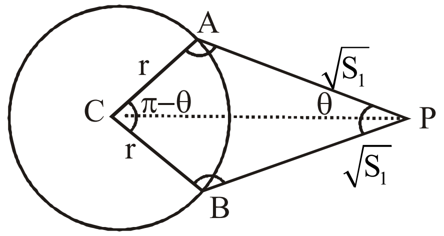
$= \frac{r}{2}$

6. Two tangents are drawn from a point P to the circle $x^2 + y^2 - 2x - 4y + 4 = 0$, such that angle between these tangents is $\tan^{-1}\left(\frac{12}{5}\right)$, where $\tan^{-1}\left(\frac{12}{5}\right) \in (0, \pi)$. If the centre of the circle is denoted by C and these tangents touch the circle at points A and B, then the ratio of the areas of ΔPAB and ΔCAB is :

(1) 11 : 4 (2) 9 : 4 (3) 3 : 1 (4) 2 : 1

Ans. Official Answer NTA (2)

Sol.



$(x-1)^2 + (y-2)^2 = 1$

$\Delta PAB = \frac{1}{2} S_1 \sin\theta$

$\Delta CAB = \frac{1}{2} r^2 \sin(\pi-\theta)$

We know $\tan\frac{\theta}{2} = \frac{r}{\sqrt{S_1}}$

$\tan\frac{\theta}{2} = \frac{1}{\sqrt{S_1}}$

$\tan\theta = \frac{12}{5}$

$\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{12}{5}$

$\frac{\frac{2}{\sqrt{S_1}}}{1-\left(\frac{1}{\sqrt{S_1}}\right)^2} = \frac{12}{5}$

hence $\sqrt{S_1} = \frac{3}{2}$

$S_1 = \frac{9}{4}$

Now $\frac{\Delta PAB}{\Delta CAB} = \frac{S_1}{r^2} = \frac{9}{4}$

7. If the curve $y = y(x)$ is the solution of the differential equation $2\left(x^2 + x^{\frac{5}{4}}\right)dy - y\left(x + x^{\frac{1}{4}}\right)dx = 2x^{\frac{9}{4}}\,dx$, $x > 0$ which passes through the point $\left(1, 1-\frac{4}{3}\log_e 2\right)$, then the value of $y(16)$ is equal to:

(1) $\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$ (2) $4\left(\frac{31}{3}+\frac{8}{3}\log_e 3\right)$ (3) $4\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$ (4) $\left(\frac{31}{3}-\frac{8}{3}\log_e 3\right)$

Ans. Official Answer NTA (3)

Sol. $2\left(x^2+x^{\frac{5}{4}}\right)dy - y\left(x+x^{\frac{1}{4}}\right)dx = 2x^{9/4}$

$$\frac{dy}{dx}-\frac{y}{2}\left(\frac{x+x^{1/4}}{x^2+x^{5/4}}\right)=\frac{x^{9/4}}{x^2+x^{5/4}}$$

$$\frac{dy}{dx}-\frac{y}{2}\left(\frac{1}{x}\right)=\frac{x^{9/4}}{x^2+x^{5/4}}$$

I. F = $e^{\int -\frac{1}{2x}dx}$

$= e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}$

$$y\left(\frac{1}{\sqrt{x}}\right)=\int\frac{1}{\sqrt{x}}\times\frac{x^{9/4}}{x^2+x^{5/4}}dx$$

$$\frac{y}{\sqrt{x}}=\int\frac{x^{7/4}dx}{x\left(x+x^{1/4}\right)}$$

$$\frac{y}{\sqrt{x}}=\int\frac{x^{3/4}}{x+x^{1/4}}\,dx$$

$x = y^4$

$dx = 4y^3dy$

$$\frac{y}{\sqrt{x}}=\int\frac{4y^3\times y^3}{y^4+y}dy$$

$$\frac{y}{\sqrt{x}}=4\int\frac{y^5\,dy}{y^3+1}$$

$y^3 = t$

$3y^2dy = dt$

$y^2dy = \frac{1}{3}\,dt$

$$\frac{y}{\sqrt{x}}=\frac{4}{3}\int\frac{t}{1+t}dt$$

$$\frac{y}{\sqrt{x}}=\frac{4}{3}\left(t-\ln(1+t)\right)+C$$

$$\frac{y}{\sqrt{x}}=\frac{4}{3}\left(y^3-\ln(1+y^3)\right)+C$$

$$\frac{y}{\sqrt{x}} = \frac{4}{3}\left(x^{\frac{3}{4}} - \ln\left(1 + x^{\frac{3}{4}}\right)\right) + C$$

It passes through $\left(1, 1 - \frac{4}{3}\ln 2\right)$

$$1 - \frac{4}{3}\ \ln 2 = \frac{4}{3}\ (1 - \ln 2) + C$$

$$C = -\frac{1}{3}$$

$$\frac{y}{\sqrt{x}} = \frac{4}{3}\left(x^{3/4} - \ln\left(1 + x^{3/4}\right)\right) - \frac{1}{3}$$

x = 16

$$\frac{y}{4} = \frac{4}{3}\left(8 - \ln(9)\right) - \frac{1}{3}$$

$$\frac{y}{4} = \frac{32}{3} - \frac{8}{3}\ln 3 - \frac{1}{3}$$

$$\frac{y}{4} = \frac{31}{3} - \frac{8}{3}\ln 3$$

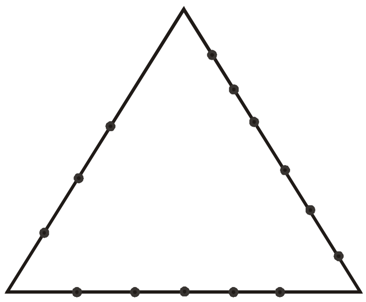
$$y = 4\left(\frac{31}{3} - \frac{8}{3}\ln 3\right)$$

8. If the sides AB, BC and CA of a triangle ABC have 3, 5 and 6 interior points respectively, then the total number of triangles that can be constructed using these points as vertices, is equal to :

(1) 333 (2) 240 (3) 364 (4) 360

Ans. Official Answer NTA (1)

Sol.



$$= {}^{14}C_3 - {}^{3}C_3 - {}^{5}C_3 - {}^{6}C_3$$

= 364 – 1 –10 – 20

= 333

9. The value of $\sum_{r=0}^{6}\left({}^{6}C_r.{}^{6}C_{6-r}\right)$ is equal to :

(1) 924 (2) 1324 (3) 1024 (4) 1124

Ans. Official Answer NTA (1)

Sol. $\sum_{r=0}^{6}{}^{6}C_r.{}^{6}C_{6-r}$

$= {}^{6}C_0 \,.\, {}^{6}C_6 + + {}^{6}C_1 .\, {}^{6}C_2 + ... + {}^{6}C_6\, {}^{6}C_0$

$= {}^{12}C_6$

$= 924$

10. Let y = y(x) be the solution of the differential equation cos x(3 sin x + cos x +3) dy = (1 + y sin x (3sin x + cos x + 3))dx, $0 \le x \le \frac{\pi}{2}$, y(0) = 0. Then, $y\left(\frac{\pi}{3}\right)$ is equal to :

(1) $2\log_e\left(\frac{2\sqrt{3}+10}{11}\right)$ (2) $2\log_e\left(\frac{\sqrt{3}+7}{2}\right)$

(3) $2\log_e\left(\frac{2\sqrt{3}+9}{6}\right)$ (4) $2\log_e\left(\frac{3\sqrt{3}-8}{4}\right)$

Ans. Official Answer NTA (1)

Sol. cos x (3 sin x + cos x + 3)dy = 1 + y sin x (3 sin x + cos x +3) dx

$$\frac{dy}{dx} + y(-\tan x) = \frac{1}{\cos x\left(3\sin x + \cos x + 3\right)}$$

$\int \tan x \; dx$

$\text{I.F.} = e^{-\int \tan x dx}$

I. F. = cos x

$y\,(\text{I. F. }) = \int (\text{I.F.})Q \; dx$

$$y\,(\cos x) = \int \frac{\cos\;x\;dx}{\cos x\left(3\sin x + \cos x + 3\right)}$$

$$y(\cos x) = \int \frac{dx}{3\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + \left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 3}$$

$$y(\cos x) = \int \frac{\sec^2\frac{x}{2}dx}{6\tan\frac{x}{2} + \left(1-\tan^2\frac{x}{2}\right) + 3\left(1+\tan^2\frac{x}{2}\right)}$$

$$y(\cos x) = \int \frac{\sec^2\frac{x}{2}dx}{2\tan^2\frac{x}{2} + 6\tan^2\frac{x}{2} + 4}$$

$$y(\cos x) = \frac{1}{2}\int \frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2} + 3\tan\frac{x}{2} + 2}$$

$$\tan\frac{x}{2} = t$$

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$y(\cos x) = \int \frac{dt}{t^2+3t+2}$$

$$y(\cos x) = \int \frac{dt}{\left(t+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}$$

$$y(\cos x) = \frac{1}{2\times\frac{1}{2}} \ln\left|\frac{2t+3-1}{2t+3+1}\right| + C$$

$$y(\cos x) = \ln\left|\frac{2t+2}{2t+4}\right| + C$$

$$y(\cos x) = \ln\left|\frac{t+1}{t+2}\right| + C$$

$$y(\cos x) = \ln\left|\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right| + C$$

x = 0 then y = 0

$$0 = \ln\left|\frac{1}{2}\right| + C$$

$$y(\cos x) = \ln\left|\frac{\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right| - \ln\frac{1}{2}$$

$$y \times \frac{1}{2} = \ln\left|\frac{\frac{1}{\sqrt{3}}+1}{\frac{1}{\sqrt{3}}+2}\right| - \ln\frac{1}{2}$$

$$\frac{y}{2} = \ln\left|\frac{1+\sqrt{3}}{1+2\sqrt{3}}\right| - \ln\frac{1}{2}$$

$$\frac{y}{2} = \ln\left|\frac{\sqrt{3}+1}{2\sqrt{3}+1}\times\frac{2\sqrt{3}-1}{2\sqrt{3}-1}\right| - \ln\frac{1}{2}$$

$$\frac{y}{2} = \ln\left|\frac{5+\sqrt{3}}{11}\right| - \ln\frac{1}{2}$$

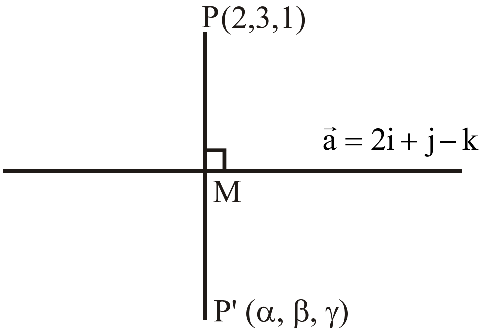
$$y = 2ln\left|\frac{10+2\sqrt{3}}{11}\right|$$

11. If the equation of plane passing through the mirror image of a point (2, 3, 1) with respect to line $\frac{x+1}{2} = \frac{y-3}{1} = \frac{z+2}{-1}$ and containing the line $\frac{x-2}{3} = \frac{1-y}{2} = \frac{z+1}{1}$ is $\alpha x + \beta y + \gamma z = 24$, then $\alpha + \beta + \gamma$ is equal to:

(1) 19 (2) 20 (3) 21 (4) 18

Ans. Official Answer NTA (1)

Sol.



M $(2\lambda - 1, \lambda + 3, -\lambda - 2)$

$P\vec{M} = (2\lambda - 3)\, i + \lambda j - (\lambda + 3)k$

$P\vec{M} \cdot \vec{a} = 0$

$2(2\lambda - 3) + \lambda(1) + (\lambda + 3) = 0$

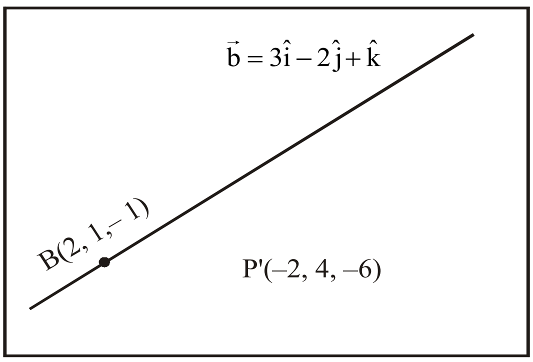
$6\lambda - 3 = 0$

$\lambda = \frac{1}{2}$

M $\left(0, \frac{7}{2}, -\frac{5}{2}\right)$

$\frac{\alpha + 2}{2} = 0,\ \frac{\beta + 3}{2} = \frac{7}{2},\ \frac{\gamma + 1}{2} = -\frac{5}{2}$

$\alpha = -2$; $\beta = 4$ $\gamma = -6$



$\overrightarrow{BP'} = -4\hat{i} + 3\hat{j} - 5\hat{k}$

Normal vector to plane $\vec{n} = \overrightarrow{BP'} \times \vec{b}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 3 & -5 \\ 3 & -2 & 1 \end{vmatrix}$$

$= \hat{i}(3 - 10) - \hat{j}(-4 + 15) + \hat{k}(-1)$

$= -7\hat{i} - 11\hat{j} - \hat{k}$

plane

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$(x\hat{i} + y\hat{j} + z\hat{k})(-7 - 11 - k) = -14 - 11 + 1$

$-7x - 11y - z = -24$

$7x + 11y + z = 24$

$\alpha x + \beta x + \gamma z = \alpha 4$

$\alpha + \beta + \gamma = 7 + 11 + 1 = 19$

12. The value of the limit $\lim\limits_{\theta \to 0} \dfrac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$ is equal to :

(1) 0 (2) $\dfrac{1}{4}$ (3) $-\dfrac{1}{2}$ (4) $-\dfrac{1}{4}$

Ans. Official Answer NTA (3)

Sol. $\lim\limits_{\theta \to 0} \dfrac{\tan(\pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$

$\lim\limits_{\theta \to 0} \dfrac{-\tan(\pi - \pi \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$

$\lim\limits_{\theta \to 0} \dfrac{-\pi(1 - \cos^2 \theta)}{\sin(2\pi \sin^2 \theta)}$

$\lim\limits_{\theta \to 0} \dfrac{-(2\pi \sin^2 \theta)}{\sin(2\pi \sin^2 \theta)} \times \dfrac{1}{2}$

$= -\dfrac{1}{2}$

13. If the Boolean expression $(p \wedge q) \circledast (p \otimes q)$ and $\circledast$ are $\otimes$ respectively given :

(1) $\rightarrow, \rightarrow$ (2) $\wedge, \vee$ (3) $\wedge, \rightarrow$ (4) $\wedge, \rightarrow$

Ans. Official Answer NTA (1)

Sol. $(p \wedge q) \circledast (p \otimes q) \Rightarrow$ is a tautolory

by option (A)

$(p \wedge q) \rightarrow (p \rightarrow q)$

p	q	p ^ q	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

14. Let $f : R \rightarrow R$ be defined as $f(x) = e^{-x} \sin x$. If $F : [0, 1] \rightarrow R$ is differential function such that $F(x) = \int_0^x f(t)\, dt$, then the value $\int_0^1 (F'(x) + f(x)) e^x\, dx$ of of lies in the inverval.

(1) $\left[\frac{330}{360}, \frac{331}{360}\right]$ (2) $\left[\frac{331}{360}, \frac{334}{360}\right]$ (3) $\left[\frac{335}{360}, \frac{336}{360}\right]$ (4) $\left[\frac{327}{360}, \frac{329}{360}\right]$

Ans. Official Answer NTA (1)

Sol. $f(x) = e^{-x} \sin^x$

$$F(x) = \int_0^x f(t)\, dt$$

$F'(x) = f(x)$

$$I = \int_0^1 \left(f(x) + f(x)\right) e^x\, dx$$

$$I = \int_0^1 2f(x) e^x\, dx$$

$$I = \int_0^1 2\left(e^{-x} \sin x\right) e^x\, dx$$

$$I = 2\int_0^1 2 \sin x\, dx$$

Now by approximation

$$x - \frac{x^3}{3!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\int_0^1\left(x-\frac{x^3}{3!}\right) \le \int_0^1 \sin x\ dx \le \int_0^1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\right)dx$$

$$\frac{11}{24} \le \int_0^1 \sin x\ dx \le \frac{331}{720}$$

$$\frac{11}{12} \le 2\int_0^1 \sin x\ dx \le \frac{331}{360}$$

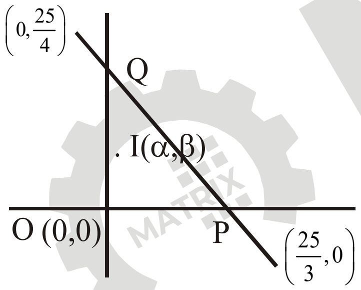
$$\frac{331}{360} \le 2\int_0^1 \sin x\ dx \le \frac{331}{360}$$

15. Let the tangent to the circle $x^2 + y^2 = 25$ at the point R(3, 4) meet x-axis and y-axis at points P and Q, respectively. If r is the radius of the circle passing through the origin O and having centre at the incentre of the triangle OPQ, then r^2 is equal to :

(1) $\frac{585}{66}$ (2) $\frac{625}{72}$ (3) $\frac{125}{72}$ (4) $\frac{529}{64}$

Ans. Official Answer NTA (2)

Sol.



tangent T = 0

3x + 4y = 25

$P\left(\frac{25}{3},0\right)$

$Q\left(0,\frac{25}{4}\right)$

$\alpha = \frac{25}{12}$

$\beta = \frac{25}{12}$

r = IO

$r = \sqrt{\frac{625}{144} + \frac{625}{144}}$

$r^2 = \frac{625}{72}$

16. Consider the function $f : R \rightarrow R$ defined by $f(x) = \begin{cases} \left(2\sin\left(\frac{1}{x}\right)\right)|x|, & x \neq 0 \\ 0 \quad , & x = 0 \end{cases}$. Then f is:

(1) monotonic on (– ∞, 0) only

(2) monotonic on (0, ∞) only

(3) not monotonic on (– ∞, 0) and (0, ∞)

(4) monotonic on (–, ∞, 0) ∪ (0, ∞)

Ans. Official Answer NTA (3)

Sol. $f(x) = \begin{cases} \left(2 - \sin\frac{1}{x}\right)x & ; \quad x > 0 \\ -\left(2 - \sin\frac{1}{x}\right)x & ; \quad x < 0 \\ 0 & ; \quad x = 0 \end{cases}$

$f'(x) = \begin{cases} \left(2 - \sin x\frac{1}{x}\right) + \frac{1}{x}\cos\frac{1}{x^2} & ; \quad x > 0 \\ -\left(2 + \sin\frac{1}{x}\right) - \frac{1}{x}\cos\frac{1}{x^2} & ; \quad x < 0 \\ 0 & ; \quad x = 0 \end{cases}$

$$f'(x) = \begin{cases} 2 - \sin x \frac{1}{x} + \frac{1}{x}\cos\frac{1}{x^2} & ; \quad x > 0 \\ -2 + \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x^2} & ; \quad x < 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Clearly here f'(x) is a oscillating function which is not monotonic in $(-\infty, 0) \cup (0, \infty)$.

17. Let O be the origin, Let $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$ and $\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$, x, y $\in$ R, x > 0, be such that $|\overrightarrow{PQ}| = \sqrt{20}$ and the vector $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OQ}$. If $\overrightarrow{OR} = 3\hat{i} + z\hat{i} - 7\hat{k}$, z $\in$ R, is coplanar with $\overrightarrow{OP}$ and $\overrightarrow{OQ}$, then the value of $x^2 + y^2 + z^2$ is equal to :

(1) 9 (2) 1 (3) 7 (4) 2

Ans. Official Answer NTA (1)

Sol. $\overrightarrow{OP} = x\hat{i} + y\hat{j} - \hat{k}$

$\overrightarrow{OQ} = -\hat{i} + 2\hat{j} + 3x\hat{k}$

$\overrightarrow{PQ} = \overrightarrow{OP} - \overrightarrow{OQ}$

$\overrightarrow{PQ} = (x+1)\hat{i} + (y-2)\hat{j} - (1+3x)\hat{k}$

$|\overrightarrow{PQ}| = \sqrt{20}$

$(x + 1)^2 + (y - 2)^2 + (3x + 1)^2 = 20$ ....(1)

$\overrightarrow{OP} \cdot \overrightarrow{OQ} = 0$

$-x + 2y - 3x = 0$

$-4x + 2y = 0$

$y = 2x$ ....(2)

by (1) & (2)

$(x + 1)^2 + (2x - 2)^2 + (3x + 1)^2 = 20$

$x = 1$ $\quad x \neq -1$

$y = 2$

also $\begin{vmatrix} 3 & z & -7 \\ 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix} = 0$

$z = -2$

$x^2 + y^2 + z^2 = 1 + 4 + 4$

$= 9$

18. The number of solutions of the equation $\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$, for $x \in [-1, 1]$ and $[x]$ denotes the greatest integer less than or equal to x, is :

(1) 4 (2) 2 (3) 0 (4) Infinite

Ans. Official Answer NTA (3)

Sol. $\sin^{-1}\left[x^2 + \frac{1}{3}\right] + \cos^{-1}\left[x^2 - \frac{2}{3}\right] = x^2$

to define $\sin^{-1}\left[x^2 - \frac{1}{3}\right]$

$-1 \le x^2 + \frac{1}{3} \le 2$

$-\frac{4}{3} \le x^2 \le \frac{5}{3}$

$0 \le x^2 \le \frac{5}{3}$ ....(1)

to define $\cos^{-1}\left[x^2 - \frac{2}{3}\right]$

$-1 \le x^2 - \frac{2}{3} < 2$

$0 \le x^2 < \frac{8}{3}$ ....(2)

by (1) & (2)

$0 \le x^2 \le \frac{5}{3}$

C -I If $0 \le x^2 < \frac{2}{3}$

$\sin^{-1}(0) + \cos^{-1}(-1) = x^2$

$x^2 = \pi$

which is not possible

C- II

$\frac{2}{3} \le x^2 \le \frac{5}{3}$

$\sin^{-1}(1) + \cos^{-1}(0) = x^2$

$\frac{\pi}{2} + \frac{\pi}{2} = x^2$

$x = \pi$

Which is also not possible hence no values of 'x' are possible.

19. If x, y, z are in arithmetic progression with common difference d, $x \ne 3d$, and the determinant of the matrix $\begin{bmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{bmatrix}$ is zero, then the value of k^2 is :

(1) 36 (2) 12 (3) 72 (4) 6

Ans. Official Answer NTA (3)

Sol. $\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 4 & 5\sqrt{2} & y \\ 5 & k & z \end{vmatrix} = 0$

$R_2 \to R_2 - \left(\frac{R_1 + R_3}{2}\right)$

$$\begin{vmatrix} 3 & 4\sqrt{2} & x \\ 0 & 5\sqrt{2} - \frac{\left(4\sqrt{2} + k\right)}{2} & 0 \\ 5 & k & z \end{vmatrix} = 0$$

Expansion by R_2

$\left(5\sqrt{2}-\frac{4\sqrt{2}+16}{2}\right)(3z-5x) = 0$

$3z - 5x = 0$

$3(x + 2\lambda) - 5x = 0$

$x \neq 3d$

$5\sqrt{2}-\frac{4\sqrt{2}+k}{2} = 0$

$10\sqrt{2} = 4\sqrt{2} + k$

$k = 6\sqrt{2}$

$k = 72$

20. Let L be a tangent line to the parabola $y^2 = 4x - 20$ at (6, 2). If L is also a tangent to the ellipse $\frac{x^2}{2}+\frac{y^2}{b} = 1$, then the value of b is equal to:

(1) 16 (2) 11 (3) 20 (4) 14

Ans. Official Answer NTA (4)

Sol. $y^2 = 4x - 20$

P (6, 2)

tangent at P'

$y = x - 4$

which is alos tangent of ellipse

$\frac{x^2}{2}+\frac{y^2}{b} = 1$

then by conditon of tangency

$c^2 = a^2m^2 + b^2$

$16 = 2 \times 1 + b$

$b = 14$

1. Let the coefficients of thirs, fourth and fifth terms in the expansion of $\left(x+\frac{a}{x^2}\right)^n$, $x \neq 0$, be in the ratio 12 : 8 : 3. Then the terms independent of x in the expansion, is equal to ______.

Ans. Official Answer NTA (4)

Sol. $T_{r+1} = {}^nC_r\ x^{n-r}\left(\frac{a}{x^2}\right)^r$

$T_{r+1} = {}^nC_r\ x^{n-3r}\ a^r$

T3 : T4 : T5

${}^nC_2\, a^2 : {}^nC_3\, a^3 : {}^nC_4\, a^4 = 12 : 8 : 3$

here by solving type we get

n = 6

$a = \frac{1}{2}$

Now term independent of x

$n - 3r = 0$

$r = 2$

$T_{2+1} = {}^6C_2\left(\frac{1}{2}\right)^2$

$= \frac{15}{4}$

So nearest integer is 4.

2. Let $I_n = \int_1^e x^{19}(\log|x|)^n\ dx$, where $n \in N$. If $(20)I_{10} = \alpha I_9 + \beta I_8$, for natural numbers α and β, then $\alpha - \beta$ equals to ___.

Ans. Official Answer NTA (1)

Sol. $I_n = \int_1^e x^{19}(\log x)^n\ dx$

$I_n = \int_1^e \underset{1}{(\log x)^n}\ \underset{2}{x^{19}}\ dx$

$I_n = \left.\frac{(\log x)^n\, x^{20}}{20}\right]_1^e - \int_1^e \frac{n(\log x)^{n-1}}{x}.\frac{x^{20}}{20}dx$

$I_n = \frac{e^{20}}{20} - \frac{n}{20}\int_1^e (\log x)^{n-1}\ x^{19}\ dx$

$20\ I_n = e^{20} - nI_{n-1}$

$20\,I_n + nI_{n-1} = e^{20}$

$n = 10$

$20\,I_{10} + 10\,I_9 = e^{20}$ ....(1)

$n = 9$

$20\,I_9 + 9\,I_8 = e^{20}$ ....(2)

By (1) & (2)

$20\,I_{10} + 10\,I_9 = 20\,I_9 + 9\,I_8$

$20\,I_{10} = 10\,I_9 + 9\,I_8$

By Comparision

$\alpha = 10$

$\beta = 9$

$\alpha - \beta = 1$

3. If 1, $\log_{10}(4^x - 2)$ and $\log_{10}\left(4^x + \frac{18}{5}\right)$ are in arithmetic progression for a real number x, then the value of the determinant $\begin{vmatrix} 2\left(x-\frac{1}{2}\right) & x-1 & x^2 \\ 1 & 0 & x \\ x & 1 & 0 \end{vmatrix}$ is equal to :

Ans. Official Answer NTA (2)

Sol. $2\log_{10}\left(4^x - 2\right) = 1 + \log_{10}\left(4^x + \frac{18}{5}\right)$

$\log_{10}\left(4^x - 2\right)^2 = \log_{10}\left(10\left(4^x + \frac{18}{5}\right)\right)$

$(4x - 2)^2 = 10.4^x + 36$ $\qquad 4^x = y$

$(y - 2)^2 = 10y + 36$

$y^2 - 4y + 4 = 10y + 36$

$y^2 - 14y - 32 = 0$

$(y - 16)(y + 2) = 0$

$y = 16$

y = –2, which is not possible

$4^x = 16$

x = 2

Now, value of determinant

$$\begin{vmatrix} 3 & 1 & 4 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix}$$

= 3(–2) – 1(0 – 4) + 4 (1)

= –6 + 4 + 4

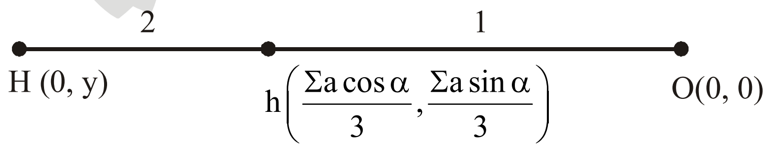
= 2

4. Let tan α, tan β and tan γ; α, β, γ $\neq \frac{(2n-1)\pi}{2}$, n ∈ N be the slopes three line segments OA, OB and OC, respectively, where O is origin. If circumcentre of ΔABC coincides with origin and its orthocentre lies on y-axis, then the value of $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\ \cos\beta\ \cos\gamma}\right)^2$ is equal to ______.

Ans. Official Answer NTA (144)

Sol. Let,

A (a cos α , a sin α)
B (a cos β , a sin β)
C (a cos γ , a sin γ)



$$\frac{2\times 0+1\times 0}{2+1} = \frac{\Sigma a\cos\alpha}{3}$$

⇒ $\Sigma\text{Cos}\,\alpha = 0$
⇒ cos α + cos β + cos γ = 0
then, $\cos^3\alpha + \cos^3\beta + \cos^3\gamma = 3\cos\alpha\ \cos\beta\ \cos\gamma$

Now, $\left(\frac{\cos 3\alpha + \cos 3\beta + \cos 3\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2$

$$\left(\frac{4(\cos^3\alpha+\cos^3\beta+\cos^3\gamma)-3(\cos\alpha+\cos\beta+\cos\gamma)}{\cos\alpha\cos\beta\cos\gamma}\right)^2$$

$$=\left(\frac{12\cos\alpha\cos\beta\cos\gamma}{\cos\alpha\cos\beta\cos\gamma}\right)^2$$

$= 144$

5. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B=\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\neq\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ such that AB = B and a + d = 2021, then the value of ad – bc is equal to _____.

Ans. Official Answer NTA (2020)

Sol. AB = B

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}=\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} a\alpha+ & b\beta \\ c\alpha+ & d\beta \end{bmatrix}=\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$a\alpha + b\beta = \gamma$

$c\alpha + d\beta = \beta$

$(a-1)\alpha + b\beta = 0$

$c\alpha + (d-1)\beta = 0$

here $\alpha = \beta \neq 0$

then $\begin{vmatrix} a-1 & b \\ c & d-1 \end{vmatrix} = 0$

$(a-1)(d-1) - bc = 0$

$ad - a - d + 1 - bc = 0$

$ad - bc = (a + d) - 1$

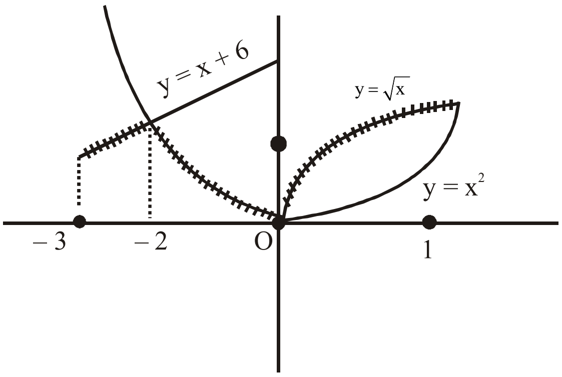
$ad - bc = 2021 - 1$

$= 2020$

6. Let $f : [-3, 1] \rightarrow R$ be given as $f(x) = \begin{cases} \min\{(x+6), x^2\} & , -3 \le x \le 0 \\ \max\{\sqrt{x}, x^2\} & , 0 \le x \le 1 \end{cases}$. If the area bounded by $y = f(x)$ and x-axis is A, then the value of 6A is equal to _________.

Ans. Official Answer NTA (41)

Sol.



$$\int_{-3}^{-2}(x+6)\,dx + \int_{-2}^{0} x^2 dx + \int_{0}^{1}\sqrt{x}\;dx$$

$$A = \frac{41}{6}$$

7. Let $f : [-1, 1] \rightarrow R$ be defined as $f(x) = ax^2 + bx + c$ for all $x \in [-1, 1]$, where $a, b, c \in R$ such that $f(-1) = 2$, $f'(-1) = 1$ and for $x \in (-1, 1)$ the maximum value of $f''(x)$ is $\frac{1}{2}$. If $f(x) \le \alpha$, $x \in [-1, 1]$, then the least value of α is equal to ________.

Ans. Official Answer NTA (5)

Sol. $f(x) = ax^2 + bx + c$

$f(-1) = a - b + c = 2$ ....(1)

$f'(x) = 2ax + b$

$f'(-1) = -2a + b = 1$ ....(2)

$f''(x) = 2a$

$$\left(f''(x)\right)_{Max} = 2a = \frac{1}{2}$$

$a = \frac{1}{4}$

they by (2) $b = 1 + 2a$

$b = 1 + \frac{1}{2}$

$b = \frac{3}{2}$

by (1) $c = 2 + b - b$

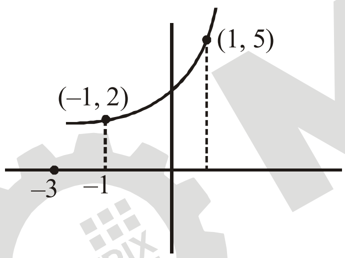
$c = 2 + - a$

$c + \frac{3}{2} - \frac{1}{4}$

$c = \frac{8+6-1}{4} = \frac{13}{4}$

$f(x) = \frac{1}{4}x^2 + \frac{3}{2}x + \frac{13}{4}$

hence



$f(x) \le \alpha$

$\alpha_{min} = 5$

8. Consider a set of 3n numbers having variance 4. In this set, the mean of first 2n numbers is 6 and the mean of the remaining n numbers is 3. A new set is constructed by adding 1 into each of first 2n numbers, and subtracting 1 from each of the remaining n numbers. If the variance of the new set is k, then 9k is equal to _______.

Ans. Official Answer NTA (68)

Sol. First αn number $\rightarrow a_1, a_2, a_3 ....a_{2n}$

Last n number $\rightarrow b_1, b_2,........b_n$

$$\sigma^2 = \frac{\sum a^2 + \sum b^2}{3n} - (\bar{x})$$ $$\bar{x} = \frac{2n \times 6 + n \times 3}{3n}$$

$$\sigma^2 = \frac{\sum a^2 + \sum b^2}{3n} - 5^2$$ $$\bar{x} = 5$$

$$\mu = \frac{\sum a^2 + \sum b^2}{3n} - 25$$

$$\sum a^2 + \sum b^2 = 87\,n$$

Now first 2n numbers $\rightarrow a_1 + 1, a_2 + 1, ...a_{2n} + 1$

Last n numbers $\rightarrow b_1 - 1, b_2 - 1, ...b_n - 1$

$$\sigma^2 = \frac{\sum (a+1)^2 + \sum (b-1)^2}{3n} - (\bar{x})^2$$

$$\sigma^2 = \sum a^2 + 2n + 2\sum a^2 +$$

$$\sigma^2 = \frac{\left(\sum a^2 + 2n + 2\sum a\right) + \left(\sum b^2 + n - 2\sum b\right)}{3n} - \left(\frac{16}{3}\right)^2$$

$$\sigma^2 = \frac{87n + 3n + 2(12n) - 2(3n)}{3n} - \left(\frac{16}{3}\right)^2$$

$$\sigma^2 = \left(\frac{108}{3}\right) - \left(\frac{16}{3}\right)^3$$

$$\sigma^2 = \frac{68}{9}$$

then Ans. $9k = 9\left(\frac{68}{9}\right) = 68$

9. Let P be an arbitrary point having sum of the squares of the distances from the planes $x + y + z = 0$, $lx - nz = 0$ and $x - 2y + z = 0$, equal to 9. If the locus of the point P is $x^2 + y^2 + z^2 = 9$, then the value of $l - n$ is equal to _____.

Ans. Official Answer NTA (0)

Sol. Let P (α, β, γ)

$d_1 + d_2 + d_3 = 9$

$$\left(\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right)^2+\left(\frac{\ell\alpha-n\gamma}{\sqrt{\ell^2+n^2}}\right)^2+\left(\frac{\alpha-2\beta+\gamma}{\sqrt{1+4+1}}\right)^2=9$$

Now Locus is

$$\left(\frac{x+y+z}{\sqrt{3}}\right)^2+\left(\frac{\ell x-nz}{\sqrt{\ell^2+n^2}}\right)^2+\left(\frac{x-2y+z}{\sqrt{6}}\right)^2=9$$

$$\left(\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}\right)+y^2\left(\frac{1}{3}+\frac{4}{6}\right)+z^2\left(\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}\right)+\alpha zx\left(\frac{1}{2}-\frac{\ell n}{\ell^2+n^2}\right)=9$$

but Locus is $x^2+y^2+z^2=9$

By comparision

$$\frac{1}{2}+\frac{\ell^2}{\ell^2+n^2}=1 \quad \text{.......................(1)}$$

$$\frac{1}{2}-\frac{\ell n}{\ell^2+n^2}=0 \quad \text{.......................(2)}$$

by (1) & (2)

$$\frac{\ell^2}{\ell^2+n^2}=\frac{1}{2}$$

$$\frac{\ell n}{\ell^2+n^2}=\frac{1}{2}$$

we get $l=n$ hence $l-n=0$

10. Let $\vec{x}$ be a vector in the plane containing vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$. If the vector is $\vec{x}$ perpendicular to $\left(3\hat{i}+2\hat{j}-\hat{k}\right)$ and its projection on $\vec{a}$ is $\frac{17\sqrt{6}}{2}$, then value of $|\vec{x}|^2$ is equal to _____.

Ans. Official Answer NTA (486)

Sol. $\vec{x}=\lambda(\vec{a})+\mu(\vec{b})$

$\vec{x}=\lambda\left(2\hat{i}-\hat{j}+k\right)+\mu\left(1+2\hat{j}-\hat{k}\right)$

$\vec{x}=\hat{i}(2\lambda+\mu)+\hat{j}(-\lambda+2\mu)+\hat{k}(\lambda-\mu)$

Now

$\vec{x}.\left(3\hat{i}+2\hat{j}-\hat{k}\right)=0$

Projectionm of $\vec{x}$ on $\vec{a}$ $= \frac{17\sqrt{6}}{2}$

$$\frac{x.\vec{a}}{|\vec{a}|} = \frac{17\sqrt{6}}{2}$$

$6\lambda - \mu = 51$ ....(2)

from (1) & (2)

$\lambda = 8$ & $\mu = -3$

$\vec{x} = 13\hat{i} - 14\hat{j} + 11\hat{k}$

$|\vec{x}|^2 = \left(\sqrt{169+196+121}\right)^2$

$|\vec{x}| = 486$