JEE Main March 2021

Question Paper With Text Solution

16 March. | Shift-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN MARCH 2021 | 16TH MARCH SHIFT-1

SECTION – A

1. If n is the number of irrational terms in the expansion of $\left(3^{1/4}+5^{1/8}\right)^{60}$, then (n –1) is divisible by :

(1) 26 (2) 30 (3) 7 (4) 8

Ans. Official Answer NTA (1)

Sol. $\left(3^{1/4}+5^{1/8}\right)^{60}$

$T_{r+1} = {}^{60}C_r \left(3^{1/4}\right)^{60-r} \left(5^{1/8}\right)^r$

$\frac{60-r}{4} = I_1$, ; $\frac{r}{8} = I_2$

$\Rightarrow r = 0, 8, 16, 24, 32, 40, 48, 56$

Total 61 terms = 8 rational terms + 53 irrational terms

$\Rightarrow$ n = 53

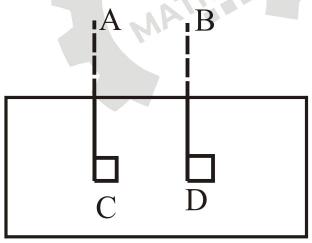
n –1 = 52

2. If for a > 0, the feet of perpendiculars from the points A(a, –2a, 3) and B(0, 4, 5) on the plane $lx + my + nz = 0$ are points C(0, –a, –1) and D respectively, then the length of line segment CD is equal to :

(1) $\sqrt{55}$ (2) $\sqrt{31}$ (3) $\sqrt{66}$ (4) $\sqrt{41}$

Ans. Official Answer NTA (3)

Sol.



$\vec{n}$ = normal vector of plane $\overrightarrow{AC} = (a, -a, -4)$

equation of plane

$ax - ay + 4z = 0$

C(0, – a, – 1) lies on it

$0 + a^2 - 4 = 0 \quad \Rightarrow \quad a = 2$

equation of plane $x - y + 2z = 0$

$C(0, -2, -1)$

equation of line BD

$\vec{r} = (0, 4, 5) + \lambda (1, -1, 2)$

$D (\lambda, 4 - \lambda, 5 + 2\lambda)$

D will lie on Plane $x - y + 2z = 0$

$\lambda - (4 - \lambda) + 2(5 + 2\lambda) = 0$

$6\lambda = -6$

$\lambda = -1$

$D(-1, 5, 3)$

$\left|\overrightarrow{CD}\right| = \sqrt{1+49+16} = \sqrt{66}$

3. The number of elements in the set $\{x \in R: (|x| -3) |x + 4| = 6\}$ is equal to :

(1) 2 (2) 1 (3) 3 (4) 4

Ans. Official Answer NTA (1)

Sol. $(|x|-3)\ |x + 4| = 6$

Case - I $x \geq 0$

$(x - 3) (x + 4) = 6$

$x^2 + x - 12 = 6$

$x^2 + x - 18 = 0$

$x = \frac{-1 \pm \sqrt{173}}{2} \Rightarrow x = \frac{1+\sqrt{73}}{2} > 0$

Case - II $-4 \leq x < 0$

$(-x - 3) (x + 4) = 6$

$x^2 + 7x + 12 - 6$

$x^2 + 7x + 18 = 0 \quad D < 0$

Case - III $x < -4$

$(-x-3)(-x-4) = 6$

$x^2 + 7x + 12 = 6$

$x^2 + 7x + 6 = 0$

$(x+1)(x+6) = 0$

$x = -1, x = -6 \quad \Rightarrow \quad x = -6$

4. The number of roots of the equation, $(81)^{\sin^2 x} + (81)^{\cos^2 x} = 30$ in the interval $[0, \pi]$ is equal to :

(1) 3 (2) 4 (3) 2 (4) 8

Ans. Official Answer NTA (2)

Sol. $81^{\sin^2 x} + 81^{1-\sin^2 x} = 30$

Put $81^{\sin^2 x} = t$

$t + \frac{81}{t} = 30$

$t^2 - 30t + 81 = 0$

$t = 27$ or $t = 3$

$81^{\sin^2 x} = 27$ $\qquad$ $81^{\sin^2 x} = 3$

$\sin^2 x = \frac{3}{4}$ $\qquad$ $\sin^2 x = \frac{1}{4}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}$ $\qquad$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$

5. If $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y \tan x = \sin x$, $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over **R** is equal to :

(1) $\frac{1}{2}$ (2) 8 (3) $-\frac{15}{4}$ (4) $\frac{1}{8}$

Ans. Official Answer NTA (4)

Sol. Linear differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$

I.f. = $e^{\int 2\tan x\, dx} = \sec^2 x$

$\int d\left(y\sec^2 x\right) = \int \sin x \sec^2 x \, dx$

$y \sec^2 x = \sec x + c$

put $x = \frac{\pi}{3}$ $y = 0$

$\Rightarrow c = -2$

$y \sec^2 x = \sec x - 2$

$y = \cos x - 2\cos^2 x$

$y_{max} = \frac{1}{8}$

6. The locus of the midpoints of the chord of the circle, $x^2 + y^2 = 25$ which is tangent to the hyperbola, $\frac{x^2}{9} - \frac{y^2}{16} = 1$ is :

(1) $(x^2 + y^2)^2 - 9x^2 - 16y^2 = 0$

(2) $(x^2 + y^2)^2 - 9x^2 + 144y^2 = 0$

(3) $(x^2 + y^2)^2 - 9x^2 + 16y^2 = 0$

(4) $(x^2 + y^2)^2 - 16x^2 + 9y^2 = 0$

Ans. Official Answer NTA (3)

Sol. Let mid point be M(h, k)

Equation of chord

$T = S_1$

$hx + ky = h^2 + k^2$

apply COT wrt hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

$c^2 = a^2m^2 - b^2$

$c^2 = 9m^2 - 16$

$\left(\frac{h^2 + k^2}{k}\right)^2 = 9\left(\frac{h^2}{k^2}\right) - 16$

$(h^2 + k^2)^2 = 9h^2 - 16k^2$

$(x^2 + y^2)^2 = 9x^2 - 16y^2$

7. Which of the following Boolean expression is a tautology ?

(1) $(p \wedge q) \vee (p \vee q)$ (2) $(p \wedge q) \wedge (p \rightarrow q)$

(3) $(p \wedge q) \rightarrow (p \rightarrow q)$ (4) $(p \wedge q) \vee (p \rightarrow q)$

Ans. Official Answer NTA (3)

Sol. (1) $(p \wedge q) \vee (p \vee q) \equiv p \vee q$

(2) $(p \wedge q) \vee (\sim p \vee q) \equiv (\sim p) \vee q$

(3) $(\sim(p \wedge q)) \vee (p \rightarrow q)$

$((\sim p) \vee (\sim q)) \vee ((\sim p) \vee q)$

$(\sim p) \vee ((\sim q) \vee q)$

$(\sim p) \vee t \equiv t$

8. If the three normals drawn to the parabola, $y^2 = 2x$ pass through the point $(a, 0)$ $a \neq 0$, then 'a' must be greater than:

(1) -1 (2) 1 (3) $\frac{1}{2}$ (4) $-\frac{1}{2}$

Ans. Official Answer NTA (2)

Sol. For parabola $y^2 = 4ax$

Equation of normal

$y = mx - 2am - am^3$

$y = mx - m - \frac{m^3}{2}$

$0 = am - m - \frac{m^3}{2}$

$m^3 = 2(a - 1)m$

$m = 0, \; m^2 = 2(a - 1)$

$\Rightarrow 2(a - 1) > 0$

$\Rightarrow a > 1$

9. If for $x \in \left(0, \frac{\pi}{2}\right)$, $\log_{10} \sin x + \log_{10} \cos x = -1$ and $\log_{10}(\sin x + \cos x) = \frac{1}{2}(\log_{10} n - 1)$, $n > 0$, then the value of n is equal to :

(1) 20 (2) 9 (3) 12 (4) 16

Ans. Official Answer NTA (3)

Sol. $\log_{10}(\sin x \cos x) = -1$

$\sin x \cos x = \frac{1}{10}$

$\sin 2x = \frac{1}{5}$

$2 \log_{10} (\sin x + \cos x) = \log_{10}\left(\frac{n}{10}\right)$

$\log_{10}(1 + \sin 2x) = \log_{10}\left(\frac{n}{10}\right)$

$1 + \sin 2x = \frac{n}{10}$

$1 + \frac{1}{5} = \frac{n}{10}$

$\Rightarrow n = 12$

10. Let the position vectors of two points P and Q be $3\hat{i} - \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} - 4\hat{k}$, respectively. Let R and S be two points such that the direction ratios of lines PR and QS are (4, –1, 2) and (–2, 1, –2), respectively. Let lines PR and QS intersect at T. If the vector $\overrightarrow{TA}$ is perpendicular to both $\overline{PR}$ and $\overline{QS}$ and the length of vector $\overrightarrow{TA}$ is $\sqrt{5}$ units, then modulus of a position vector of A is :

(1) $\sqrt{171}$ (2) $\sqrt{227}$ (3) $\sqrt{482}$ (4) $\sqrt{5}$

Ans. Official Answer NTA (1)

Sol. Equation of line

PR $\vec{r} = (3, -1, 2) + \lambda (4, -1, 2)$

QS : $\vec{r} = (1, 2, -4) + \mu (-2, 1, -2)$

POI $\equiv$ T

$3 + 4\lambda = 1 - 2\mu$

$-1 - \lambda = 2 + \mu \qquad \Rightarrow \lambda = 2,\ \mu = -5$

$2 + 2\lambda = -4 - 2\mu$

T(11, –3, 6)

$\vec{n}$ = vector parallel to $\overrightarrow{TA}$

$$= \overrightarrow{PR} \times \overrightarrow{QS} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 2 \\ -2 & 1 & -2 \end{vmatrix}$$

$= 4\hat{j} + 2\hat{k}$

equaiton of $\overrightarrow{TA}$: $\vec{r} = (11, -3, 6) + \lambda\,(0, 2, 1)$

$\left|\overrightarrow{TA}\right| = \sqrt{5} \Rightarrow \lambda = \pm 1$

A (11, –1, 7) or A(11, –5, 5)

$\left|\overrightarrow{OA}\right| = \sqrt{121+1+49} = \sqrt{171}$

11. Let $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$, $i = \sqrt{-1}$. Then the system of linear equations $A^8 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 64 \end{bmatrix}$ has:

(1) Infinitely many solutions (2) Exactly two solutions

(3) A unique solution (4) No solution

Ans. Official Answer NTA (4)

Sol. $A^2 = AA = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}\begin{bmatrix} i & -i \\ -i & i \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

$A^4 = A^2A^2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$

$A^8 = A^4A^4 = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}\begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} = \begin{bmatrix} 128 & -128 \\ -128 & 128 \end{bmatrix}$

$128x - 128y = 8$

$-128x + 128y = 64$

system will have no solution

12. The range of $a \in R$ for which the function $f(x) = (4a-3)(x + \log_e 5) + 2(a-7)\cot\left(\frac{x}{2}\right)\sin^2\left(\frac{x}{2}\right)$, $x \neq 2n\pi$, $n \in N$ has critical points, is :

(1) $(-3, 1)$ (2) $\left[-\frac{4}{3}, 2\right]$ (3) $(-\infty, -1]$ (4) $[1, \infty)$

Ans. Official Answer NTA (2)

Sol. $f'(x) = (4a-3) + (a-7)\cos x$

Equation $f'(x) = 0$ has atleast one solution

Range of $f'(x) \equiv [3a+4, 5a-10]$

$(3a+4)(5a-10) \leq 0$

$a \in \left[-\frac{4}{3}, 2\right]$

13. A pack of cards has one card missing. Two cards are drawn randomly and are found to be spades. The probability that the missing card is not a spade, is :

(1) $\frac{22}{425}$ (2) $\frac{3}{4}$ (3) $\frac{52}{867}$ (4) $\frac{39}{50}$

Ans. Official Answer NTA (4)

Sol. $E_1 \equiv$ Missing card is spades

$E_2 \equiv$ Missing card is not spades

$A \equiv$ Two cards are found to be spades

$$p\left(\frac{E_2}{A}\right) = \frac{p(E_2 \cap A)}{p(E_1 \cap A) + p(E_2 \cap A)}$$

$$= \frac{\frac{3}{4} \times \frac{^{13}C_2}{^{51}C_2}}{\frac{1}{4} \times \frac{^{12}C_2}{^{51}C_2} + \frac{3}{4} \times \frac{^{13}C_2}{^{51}C_2}}$$

$$= \frac{3\left(^{13}C_2\right)}{^{12}C_2 + 3\left(^{13}C_2\right)} = \frac{234}{66 + 234}$$

$= \frac{234}{300} = \frac{39}{50}$

14. Consider three observations a, b and c such that b = a + c. If standard deviation of a + 2, b + 2, c + 2 is d, then which of the following is true ?

(1) $b^2 = a^2 + c^2 + 3d^2$ (2) $b^2 = 3(a^2 + c^2) - 9d^2$

(3) $b^2 = 3(a^2 + c^2) + 9d^2$ (4) $b^2 = 3(a^2 + c^2 + d^2)$

Ans. Official Answer NTA (2)

Sol. $AM = \bar{x} = \frac{a+b+c}{3} = \frac{2b}{3}$

$$\sigma = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

$$d^2 = \left(\frac{a^2+b^2+c^2}{3}\right) - \left(\frac{2b}{3}\right)^2$$

$9d^2 = 3(a^2 + b^2 + c^2) - 4b^2$

$b^2 = 3a^2 + 3c^2 - 9d^2$

15. Let $S_k = \sum_{r=1}^{k} \tan^{-1}\left(\frac{6^r}{2^{2r+1} + 3^{2r+1}}\right)$. Then $\lim_{k\to\infty} S_k$ is equal to :

(1) $\cot^{-1}\left(\frac{3}{2}\right)$ (2) $\tan^{-1}\left(\frac{3}{2}\right)$ (3) $\tan^{-1}(3)$ (4) $\frac{\pi}{2}$

Ans. Official Answer NTA (1)

Sol. $$T_r = \tan^{-1}\left(\frac{\frac{6^r}{2^{2r+1}}}{1 + \frac{3^{2r+1}}{2^{2r+1}}}\right)$$

$$= \tan^{-1}\left(\frac{\left(\frac{3}{2}\right)^{r+1} - \left(\frac{3}{2}\right)^{r}}{1 + \left(\frac{3}{2}\right)^{r+1}\left(\frac{3}{2}\right)^{r}}\right)$$

$T_r = \tan^{-1}\left(\frac{3}{2}\right)^{r+1} - \tan^{-1}\left(\frac{3}{2}\right)^{r}$

$S_k = \tan^{-1}\left(\frac{3}{2}\right)^{k+1} - \tan^{-1}\left(\frac{3}{2}\right)$

$k \to \infty$

$\lim_{k\to\infty} S_k = \frac{\pi}{2} - \tan^{-1}\left(\frac{3}{2}\right) = \cot^{-1}\left(\frac{3}{2}\right)$

16. Let a vector $\alpha\hat{i} + \beta\hat{j}$ be obtained by rotating the vector $\sqrt{3}\hat{i} + \hat{j}$ by an angle 45º about the origin in counterclockwise direction in the first quadrant. Then the area of triangle having vertices (α,β), $(0,\beta)$ and (0,0) is equal to:

(1) $\frac{1}{\sqrt{2}}$ (2) 1 (3) $2\sqrt{2}$ (4) $\frac{1}{2}$

Ans. Official Answer NTA (4)

Sol. $\alpha\hat{i} + \beta\hat{j} = 2\left(\cos 75^\circ\,\hat{i} + \sin 75^\circ\,\hat{j}\right)$

Area $= \frac{1}{2}\left\|\begin{vmatrix} \alpha & \beta & 1 \\ 0 & \beta & 1 \\ 0 & 0 & 1 \end{vmatrix}\right\| = \left|\frac{1}{2}\alpha\beta\right|$

$= \frac{1}{2}\left|2.2\sin 75^\circ \cos 75^\circ\right| = \frac{2\sin 150^\circ}{2} = \frac{1}{2}$

17. Let a complex number $z, |z| \neq 1$, satisfy $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$. Then the largest value of $|z|$ is equal to ____.

(1) 6 (2) 8 (3) 5 (4) 7

Ans. Official Answer NTA (4)

Sol. $\log_{\frac{1}{\sqrt{2}}}\left(\frac{|z|+11}{(|z|-1)^2}\right) \leq 2$

Let $|z| = t$

$$\frac{t+11}{(t-1)^2} \geq \frac{1}{2}$$

$2t + 22 \geq (t-1)^2$

$t^2 - 4t - 21 \leq 0$

$(t-7)(t+3) \leq 0$

$\Rightarrow t \leq 7$

$|z| \leq 7$

18. Let the functions $f : R \to R$ and $g : R \to R$ be defined as :

$$f(x) = \begin{cases} x+2 & x<0 \\ x^2 & x \geq 0 \end{cases} \text{ and } g(x) = \begin{cases} x^3, & x<1 \\ 3x-2, & x \geq 1 \end{cases}$$

Then, the number of points in R where (fog) (x) is NOT differentiable is equal to :

(1) 2 (2) 1 (3) 3 (4) 0

Ans. Official Answer NTA (2)

Sol. Check points will x = 1 & x = 0

$$f(x) = \begin{cases} (3x-2)^2 & x>1 \\ x^6 & 0 \leq x < 1 \\ x^3+2 & x<0 \end{cases}$$

at x = 1

LHD = 6

RHD = 6; f(x) is derivable at x = 1

at x = 0

LHD = DNE

RHD = 0 ; f(x) is not derivable at x = 1

19. Let [x] denote greatest integer less than or equal to x. If for $n \in N$, then $(1 - x + x^3)^n = \sum_{j=0}^{3n} a_j x^j$, then

$$\sum_{j=0}^{\left[\frac{3n}{2}\right]} a_{2j} + 4\sum_{j=0}^{\left[\frac{3n-1}{2}\right]} a_{2j+1}$$ is equal to :

(1) n (2) 2^{n-1} (3) 2 (4) 1

Ans. Official Answer NTA (4)

Sol. $f(x) = (1 - x + x^3)^n$

$f(1) = 1, f(-1) = 1$

$$\sum a_{2j} = \frac{f(1)+f(-1)}{2} = \frac{1+1}{2} = 1$$

$$\sum a_{2j+1} = \frac{f(1)-f(-1)}{2} = \frac{1-1}{2} = 0$$

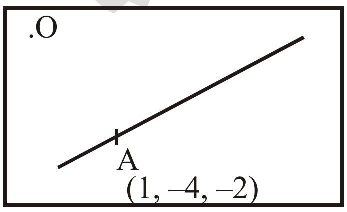
Required sum = 1 + 4 (0) = 1

20. Let P be a plane $lx + my + nz = 0$ containing the line, $\frac{1-x}{1} = \frac{y+4}{2} = \frac{z+2}{3}$. If plane P divides the line segment AB joining points A(– 3, – 6, 1) and B(2, 4, – 3) in ratio k : 1 then the value of k is equal to :

(1) 4 (2) 2 (3) 1.5 (4) 3

Ans. Official Answer NTA (2)

Sol.



$$\vec{n} = \overrightarrow{OA} \times (-1, 2, 3)$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -2 \\ -1 & 2 & 3 \end{vmatrix}$$

$$= 8\hat{i} - \hat{j} - 2\hat{k}$$

Equation of plane

$8x + y + 2z = 0$

Let POI be M

$M\left(\frac{2k-3}{k+1}, \frac{4k-6}{k+1}, \frac{1-3k}{k+1}\right)$

M will lie m plane P

$8\left(\frac{2k-3}{k+1}\right) + \frac{4k-6}{k+1} + 2\left(\frac{1-3k}{k+1}\right) = 0$

$16k - 24 + 4k - 6 + 2 - 6k = 0$

$14k = 28$

$k = 2$

Section - B

1. Let the crve $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx} = 2(x + 1)$. If the numerical value of area bounded by the curve $y = y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to ______.

Ans. Official Answer NTA (2)

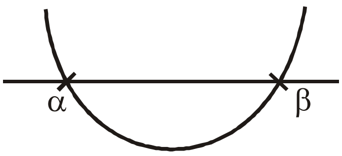
Sol.

$\frac{dy}{dx} = 2(x + 1)$

$\int dy = \int (2x + 2) dx$

$y = x^2 + 2x + c$

$\alpha < \beta$ are roots of equation $x^2 + 2x + c = 0$



$\int_{\alpha}^{\beta} -\left(x^2 + 2x + c\right)\, dx = \frac{4\sqrt{8}}{3}$

$$\frac{(\beta-\alpha)^3}{6} = \frac{4\sqrt{8}}{3}$$

$(\beta - \alpha)^3 = 8\sqrt{8}$

$(\beta - \alpha) = \sqrt{8}$

$\sqrt{4-4c} = \sqrt{8}$

$4 - 4c = 8 \Rightarrow c = -1$

$f(x) = x^2 + 2x - 1$

$f(1) = 1 + 2 - 1 = 2$

2. If $\lim\limits_{x\to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$, then a + b + c is equal to _______.

Ans. Official Answer NTA (4)

Sol. $\lim\limits_{x\to 0} \dfrac{ae^x + ce^{-x} + b\cos x}{x\sin x} = 2$

$$\lim_{x\to 0} \frac{a\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}.....\right)+c\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}......\right)-b\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}.......\right)}{x^2} = 2$$

$\Rightarrow$ Coefficient of $x^0 = 0$

$a + c - b = 0$

$\Rightarrow$ Coefficient of $x^1 = 0$

$a - c = 0$

$\Rightarrow$ Coefficient of $x^2 = 2$

$\frac{a}{2} + \frac{c}{2} + \frac{b}{2} = 2$

$a + b + c = 4$

3. Let $f: R \to R$ be a continuous function such that $f(x) + f(x + 1) = 2$, for all $x \in R$. If $I_1 = \int_0^8 f(x)\,dx$ and $I_2 = \int_{-1}^{3} f(x)\,dx$, then the value of $I_1 + 2I_2$ is equal to ______.

Ans. Official Answer NTA (16)

Sol. $f(x) + f(x + 1) = 2$

$x \to x + 1$

$f(x + 1) + f(x + 2) = 2$

$\Rightarrow f(x) = f(x + 2)$

Period of f(x) is 2.

$I = \int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$

$x = t + 1$

$I = \int_0^1 f(x)\,de + \int_0^1 f(x)\,dt$

$I = 2$

$I_1 = \int_0^8 f(x)\,dx = 4\int_0^2 f(x)\,dx = 8$ ($\because$ f(x) is periodic with period 2)

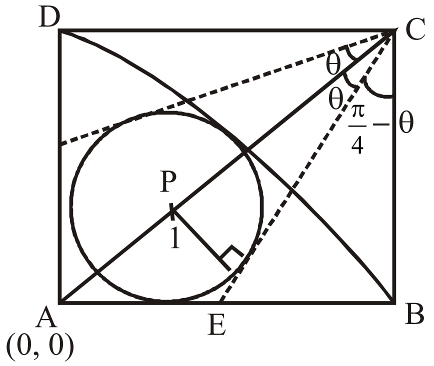
$I_2 = \int_{-1}^3 f(x)\,dx = 2\int_0^2 f(x)\,dx = 4$ ($\because$ f(x) is periodic with period 2)

$I_1 + 2I_2 = 8 + 8 = 16$

4. Let ABC be a square of side of unit length. Let a circle C_1 centered at A with unit radius is drawn. Another circle C_2 which touches C_1 and the lines AD and AB are tangent to it, is also drawn. Let a tangent line from the point C to the circle C_2 meet the side AB at E. If the length of EB is $\alpha + \sqrt{3}\beta$, where α, β are integers, then $\alpha + \beta$ is equal to _____.

Ans. Official Answer NTA (1)

Sol.



Let centre of C_2 = P (r, r)

$\sqrt{2}\,r = |1 - r|$

$\sqrt{2}\,r = 1 - r$

$(\sqrt{2} + 1)r = 1$

$r = \sqrt{2} - 1$

PC = AC – AP

$PC = AC - \sqrt{2}\,r$

$= \sqrt{2} - \sqrt{2}\,(\sqrt{2} - 1)$

$PC = 2(\sqrt{2} - 1)$

$$\sin\theta = \frac{r}{PC} = \frac{\sqrt{2}-1}{2\left(\sqrt{2}-1\right)} = \frac{1}{2}$$

$\theta = 30^\circ$

$$\frac{BE}{BC} = \tan\left(\frac{\pi}{4} - \theta\right)$$

$BE = \tan(15^\circ) = 2 - \sqrt{3}$

$\alpha = 2,\ \beta = -1$

$\alpha + \beta = 1$

5. Consider an arithmetic series and a geometric series having four intial terms from the set {11, 8, 21, 16, 26, 32, 4}. If the last terms of these series are the maximum possible four digit numbers, then the number of common terms in these two series is equal to _____.

Ans. Official Answer NTA (3)

Sol. G.P. $\equiv \{4, 8, 16, 32 \ldots\ldots\}$

A.P. $\equiv \{11, 16, 21, 26 \ldots\ldots\}$

G.P. will end at 8192

All terms of GP whose units digit is 6 will also be part of A.P.

Common terms $\equiv \{16, 256, 4096\}$

Ans = 3

6. The total number of 3 × 3 matrices A having entries from the set {0, 1, 2, 3} such that the sum of all the diagonal entries of AA^T is 9, is equal to ______.

Ans. Official Answer NTA (766)

ol. $A = \begin{bmatrix} a_a & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

Sum of diagonal entries of $AA^T = \sum a_i^2 + \sum b_i^2 + \sum c_1^2$

$\{0, 0, 0, \ldots\ldots\ldots\ldots 0, 3\} \Rightarrow \frac{\underline{|9}}{\underline{|8}} = 9$

$\{0, 0, 0, 1, 1, 1, 1, 1, 2\} \frac{\underline{|9}}{\underline{|3}\,\underline{|5}\,\underline{|1}} = 504$

$\{2, 2, 1, 0, 0, 0, 0, 0, 0\} \frac{\underline{|9}}{\underline{|2}\,\underline{|6}} = 252$

$\{1, 1, 1, 1, 1, 1, 1, 1, 1\} \frac{\underline{|9}}{\underline{|9}} = 1$

Total = 9 + 504 + 252 + 1 = 766

7. Let z and w be two complex numbers such that $w = z\bar{z} - 2z + 2$, $\left|\frac{z+i}{z-3i}\right| = 1$ and Re (w) has minimum value. Then, the minimum value of $n \in N$ for which w^n is real, is equal to _____.

Ans. Official Answer NTA (4)

Sol. $z = x + iy$

$\left|\dfrac{z+i}{z-3i}\right| = 1$

$|z + i| = |z - 3i|$

$|x + iy + i| = |x + y - 3i|$

$\Rightarrow y = 1$

$w = z\bar{z} - 2z + 2$

$w = x^2 - 2x + 3 - 2i$

Re $(w) = x^2 - 2x + 3$ will be minimum when x = 1

$\Rightarrow w = 2 - 2i \;\; = 2\sqrt{2}\; e^{-i\frac{\pi}{4}}$

$w^n = (2\sqrt{2})^n \;\; e^{-i\frac{n\pi}{4}}$

$\Rightarrow \dfrac{n\pi}{4} = k\pi$

$n = 4k$

$n|_{min.} = 4$

8. If the normal to the curve $y(x) = \int_0^x \left(2t^2 - 15t + 10\right) dt$ at a point (a, b) is parallel to the line x + 3y = –5, a > 1, then the value of |a + 6b| is equal to ______.

Ans. Official Answer NTA (406)

Sol. $\dfrac{dy}{dx} = 3$

$2x^2 - 15x + 10 = 3$

$2x^2 - 15x + 7 = 0$

$(x - 7)(2x - 1) = 0$

$\Rightarrow x = 7 \;\; (\because\; x > 1) \;\; \Rightarrow a = 7$

$b = \int_0^7 \left(2t^2 - 15t + 10\right) dt$

$b = \left(\dfrac{2t^3}{3} - \dfrac{15t^2}{2} + 10t\right)_0^7$

$= \frac{686}{3} - \frac{735}{2} + 70$

$= \frac{1372 - 2205 + 420}{6}$

$b = \frac{-413}{6}$

$|a + 6b| = |7 - 413| = 406$

9. Let $f : (0, 2) \rightarrow \mathbf{R}$ be defined as $f(x) = \log_2\left(1 + \tan\left(\frac{\pi x}{4}\right)\right)$. Then, $\lim_{n\to\infty} \frac{2}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + ... + f(1)\right)$ is equal to _____.

Ans. Official Answer NTA (1)

Sol. $L = \lim_{n\to\infty} \frac{2}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) .... + f(1)\right)$

$= 2\int_0^1 f(x)dx$

$L = 2\int_0^1 \log_2\left(1 + \tan\frac{\pi x}{4}\right)dx$ ....(1)

Apply king

$L = 2\int_0^1 \log_2\left(1 + \tan\frac{\pi(1-x)}{4}\right)dx$ ....(2)

(1) + (2)

$2L = 2\int_0^1 \log_2(2)dx = 2$

$L = 1$

10. Let $P = \begin{bmatrix} -30 & 20 & 56 \\ 90 & 140 & 112 \\ 120 & 60 & 14 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 7 & \omega^2 \\ -1 & -\omega & 1 \\ 0 & -\omega & \omega+1 \end{bmatrix}$ where $\omega = \frac{-1 + i\sqrt{3}}{2}$, and I_3 be the identity matrix of order 3. If the determinant of the matrix $(P^{-1}AP - I_3)^2$ is $\alpha\omega^2$, then the value of α is equal to ______.

Ans. Official Answer NTA (36)

Sol. $M = (P^{-1} AP - I)(P^{-1} AP - I)$

$= P^{-1} AP P^{-1} AP - P^{-1} AP - P^{-1} AP + I$

$= P^{-1} A^2P - 2P^{-1} AP + P^{-1} IP$

$= P^{-1} (A^2 - 2A + I)P$

$M = P^{-1} (A - I)^2 P$

$|M| = |P^{-1}|\,|A - I|^2\,|P|$

$= |A - I|^2$

$$|A - I|^2 = \begin{vmatrix} 1 & 7 & \omega^2 \\ -1 & -1-\omega & 1 \\ 0 & -\omega & -\omega \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & 7 & \omega^2 \\ -1 & \omega^2 & 1 \\ 0 & -\omega & -\omega \end{vmatrix}^2$$

$= \left(-1 + \omega + 7(-\omega) + \omega^2 (w)\right)^2$

$= (-6\omega)^2 = 36\omega^2$

$\Rightarrow \alpha = 36$