JEE Main September 2020

Question Paper With Text Solution

4 September| Shift-1

MATHEMATICS

JEE MAIN SEP 2020 | 4 SEP SHIFT-1

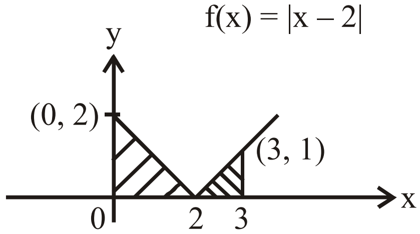
1. Let f(x) = |x – 2| and g(x) = f(f(x)), $x \in [0,4]$. Then $\int_0^3 (g(x) - f(x))dx$ is equal to :

(1) 0 (2) $\frac{1}{2}$ (3) $\frac{3}{2}$ (4) 1

Ans (4)

S. f(x) = |x – 2| g(x) = f (|x – 2|) = ||x – 2| –2|

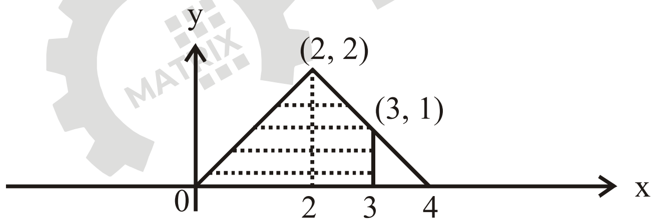
f(x) = |x – 2|



$\int_0^3 f(x)dx =$ Area of shaded region

$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 1 \times 1 = \frac{5}{2}$

g(x) = ||x – 2| – 2|



$\int_0^3 g(x)dx =$ Area of shaded region

$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2}(1).(1+2) = \frac{7}{2}$

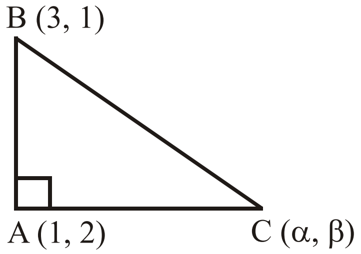
$\therefore \int_0^3 (g(x) - f(x))dx = \int_0^3 g(x)dx - \int_0^3 f(x)dx = \frac{7}{2} - \frac{5}{2} = 1$

2. A triangle ABC lying in the first quadrant has two vertices as A(1, 2) and B(3, 1). If $\angle BAC = 90°$, and $ar(\Delta ABC) = 5\sqrt{5}$ sq. units, then the abscissa of the vertex C is :

(1) $1+\sqrt{5}$ (2) $2+\sqrt{5}$ (3) $2\sqrt{5}-1$ (4) $1+2\sqrt{5}$

Ans (4)

Sol.



$m_{AB} \times m_{AC} = -1$

$\Rightarrow \frac{\beta-2}{\alpha-1}.\frac{2-1}{1-3} = -1$

$\Rightarrow \frac{\beta-2}{\alpha-1}.\left(-\frac{1}{2}\right) = -1$

$\Rightarrow \frac{\beta-2}{\alpha-1} = 2$

$\Rightarrow \beta - 2 = 2\alpha - 2$

$\beta = 2\alpha$ .... (1)

Area of $\Delta ABC = 5\sqrt{5}$

$\frac{1}{2} \times AB \times AC = 5\sqrt{5}$

$\Rightarrow AB^2 \times AC^2 = 125 \times 4$

$\left((\alpha-1)^2 + (\beta-2)^2\right)\left((1-3)^2 + (2-1)^2\right) = 500$

$\left((\alpha-1)^2 + 4(\alpha-1)^2\right)(4+1) = 500$

$\Rightarrow 5(\alpha-1)^2.5 = 500$

$(\alpha-1)^2 = 20$ $|\alpha-1| = 2\sqrt{5}$ $\therefore \alpha = 1 \pm 2\sqrt{5}$

$\therefore \alpha = 1+2\sqrt{5}$, As ΔABC lying in first quadrant.

3. Let $y = y(x)$ be the solution of the differential equation, $xy' - y = x^2\left(x\cos x + \sin x\right), x > 0$. If $y\left(\pi\right) = \pi$, then $y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right)$ is equal to :

(1) $2+\frac{\pi}{2}+\frac{\pi^2}{4}$ (2) $1+\frac{\pi}{2}+\frac{\pi^2}{4}$ (3) $2+\frac{\pi}{2}$ (4) $1+\frac{\pi}{2}$

Ans (3)

Sol. $x\,y' - y = x^2\,(x\cos x + \sin x)$

$y' - \frac{1}{x}\,y = x\,(x\cos x + \sin x)$

$\frac{dy}{dx} - \frac{1}{x}y = x\,(x\cos x + \sin x)$

If $= e^{\int -\frac{1}{x}dx}$

$= e^{-\ln x} = \frac{1}{x}$

$y \cdot \frac{1}{x} = \int \frac{1}{x}.x\,(x\cos x + \sin x)\,dx + c$

$\Rightarrow \frac{y}{x} = \int (x\cos x + \sin x)\,dx + c$

$\Rightarrow \frac{y}{x} = x\sin x - \int \sin x\,dx - \cos x + c$

$\Rightarrow \frac{y}{x} = x\sin x + \cos x - \cos x + c$

$y = x^2 \sin x + cx$

$y(x) = x^2 \sin x + cx$

$\therefore\; y(\pi) = 0 + c\pi = \pi$

$\therefore\; c = 1$

$y(x) = x^2 \sin x + x$

$y'(x) = 2x \sin x + x^2 \cos x + 1$

$y''(x) = 2\sin x + 2x \cos x + 2x \cos x - x^2 \sin x$

$= 2\sin x + 4x \cos x - x^2 \sin x$

$\therefore\; y\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} + \frac{\pi}{2}, \quad y''\left(\frac{\pi}{2}\right) = 2 - \frac{\pi^2}{4}$

$\Rightarrow y''\left(\frac{\pi}{2}\right)+y\left(\frac{\pi}{2}\right)=2+\frac{\pi}{2}$

4. The value of $\sum_{r=0}^{20} {}^{50-r}C_6$ is equal to :

(1) ${}^{50}C_7 - {}^{30}C_7$ (2) ${}^{51}C_7 - {}^{30}C_7$ (3) ${}^{51}C_7 + {}^{30}C_7$ (4) ${}^{50}C_6 - {}^{30}C_6$

Ans (2)

Sol. $\sum_{r=0}^{20} {}^{50-r}C_6$

$= {}^{50}C_6 + {}^{49}C_6 + ... + {}^{30}C_6$

$=$ Coefficient of x^6 in $\left[(1+x)^{30}+(1+x)^{31}+....+(1+x)^{50}\right]$

$=$ Coefficient of x^6 in $\left[\frac{(1+x)^{30}\left((1+x)^{21}-1\right)}{(1+x)-1}\right]$

$=$ Coefficient of x^6 in $\left[\frac{(1+x)^{51}-(1+x)^{30}}{x}\right]$

$= {}^{51}C_7 - {}^{30}C_7$

5. Given the following two statements :

$(S_1):(q \vee p) \rightarrow (p \leftrightarrow \sim q)$ is a tautology.

$(S_2):\sim q \wedge (\sim p \leftrightarrow q)$ is a fallacy. Then :

(1) both (S_1) and (S_2) are correct

(2) both (S_1) and (S_2) are not correct

(3) only (S_2) is correct

(4) only (S_1) is correct

Ans (2)

Sol.

p	q	$\sim p$	$\sim q$	$q \vee p$	$p \leftrightarrow \sim q$	$(q \vee p) \to (p \leftrightarrow \sim q)$	$\sim p \leftrightarrow q$	$\sim q \wedge (\sim p \leftrightarrow q)$
T	T	F	F	T	F	F	F	F
T	F	F	T	T	T	T	T	T
F	T	T	F	T	T	T	T	F
F	F	T	T	F	F	T	F	F

S_1: $(q \vee p) \to (p \leftrightarrow \sim q)$ ↓ Not a tautology

S_2: $\sim q \wedge (\sim p \leftrightarrow q)$ ↓ Not a fallacy

6. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1(a>b)$ be a given ellipse, length of whose latus rectum is 10. If its eccentricity is the maximum value of the function, $\phi(t)=\frac{5}{12}+t-t^2$, then a^2+b^2 is equal to :

(1) 126 (2) 135 (3) 145 (4) 116

Ans (1)

Sol. Maximum value of $\phi(t)=-\frac{D}{4a}$

$$=\frac{-\left[1+4.\frac{5}{12}\right]}{-4}$$

$$=\frac{1}{4}\left[\frac{8}{3}\right]=\frac{2}{3}$$

$\therefore$ Eccentricity of ellipse = $\phi(t)\big|_{max}=\frac{2}{3}$

We have

Latus Rectum = 10

$$\frac{2b^2}{a}=10$$

$\therefore$ $b^2 = 5a$ .... (1)

$$e=\frac{2}{3}$$

$e^2 = \frac{4}{9}$

$\Rightarrow 1 - \frac{b^2}{a^2} = \frac{4}{9}$

$\Rightarrow 1 - \frac{5a}{a^2} = \frac{4}{9}$

$\Rightarrow 1 - \frac{4}{9} = \frac{5}{a}$

$\therefore$ a = 9, $\therefore$ $b^2 = 45$

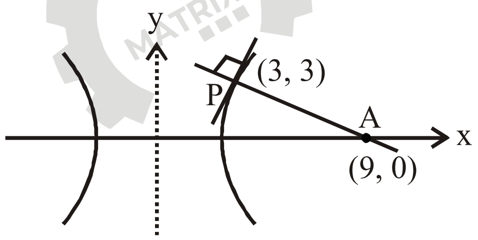
$\because$ $a^2 + b^2 = 81 + 45 = 126$

7. Let P(3, 3) be a point on the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If the normal to it at P intersects the x–axis at (9, 0) and e is its eccentricity, then the ordered pair (a^2, e^2) is equal to :

(1) $\left(\frac{9}{2}, 2\right)$ (2) $\left(\frac{3}{2}, 2\right)$ (3) $\left(\frac{9}{2}, 3\right)$ (4) (9, 3)

Ans (3)

Sol.



H : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\therefore$ P lies on it

$\therefore$ $\frac{9}{a^2} - \frac{9}{b^2} = 1$

$\therefore$ $\frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{9}$ .... (1)

Equation of tangent at P :

$$\frac{x.3}{a^2}-\frac{y.3}{b^2}-1=0$$

Slope of tangent at $P=\frac{b^2}{a^2}$

$\therefore$ Slope of normal at $P=-\frac{a^2}{b^2}$

$$\frac{3-0}{3-9}=-\frac{a^2}{b^2}$$

$$\Rightarrow \frac{1}{2}=\frac{a^2}{b^2} \text{ .... (2)}$$

$$\frac{1}{a^2}-\frac{1}{2a^2}=\frac{1}{9}$$

$$\Rightarrow \frac{1}{2a^2}=\frac{1}{9}$$

$$a^2=\frac{9}{2},\quad b^2=9$$

$$e^2=1+\frac{b^2}{a^2}$$

$$=1+\frac{9}{\frac{9}{2}}$$

$= 1 + 2 = 3$

$(a^2, e^2) = \left(\frac{9}{2},3\right)$

8. Let x_0 be the point of local maxima of $f(x)=\vec{a}.(\vec{b}\times\vec{c})$, where $\vec{a}=x\hat{i}-2\hat{j}+3\hat{k}$, $\vec{b}=-2\hat{i}+x\hat{j}-\hat{k}$ and $\vec{c}=7\hat{i}-2\hat{j}+x\hat{k}$. Then the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$ at $x = x_0$ is :

(1) –4 (2) –22 (3) –30 (4) 14

Ans (2)

S. $f(x) = [\vec{a}\vec{b}\vec{c}]$

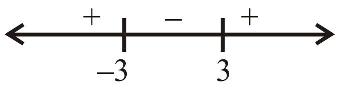
$$= \begin{vmatrix} x & -2 & 3 \\ -2 & x & -1 \\ 7 & -2 & x \end{vmatrix}$$

Expand

$f(x) = x^3 - 27x + 26$

$f'(x) = 3x^2 - 27 = 0$

$\therefore x = \pm 3$



x = –3 is the point of local maxima.

$\therefore\ x_0 = -3$ .... (1)

$\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

$= 3x - 13$

at $x = x_0 = 3x_0 - 13$

$= 3(-3) - 13 = -22$

9. If $A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}, \left(\theta = \frac{\pi}{24}\right)$ and $A^5 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $i = \sqrt{-1}$, then which one of the following is not true?

(1) $a^2 - d^2 = 0$ (2) $a^2 - b^2 = \frac{1}{2}$ (3) $0 \le a^2 + b^2 \le 1$ (4) $a^2 - c^2 = 1$

Ans (2)

Sol. $A^2 = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + i^2\sin^2\theta & i(2\cos\theta\sin\theta) \\ 2i\sin\theta.\cos\theta & \cos^2\theta + i^2\sin^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin^2\theta & \cos 2\theta \end{bmatrix}$$

$A^3 = \begin{bmatrix} \cos 2\theta & i\sin 2\theta \\ i\sin 2\theta & \cos 2\theta \end{bmatrix}\begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$

$= \begin{bmatrix} \cos 3\theta & i\sin 3\theta \\ i\sin 3\theta & \cos 3\theta \end{bmatrix}$

$A^5 = \begin{bmatrix} \cos 5\theta & i\sin 5\theta \\ i\sin 5\theta & \cos 5\theta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a = \cos5\theta \qquad b = i\sin5\theta$

$c = i\sin5\theta \qquad d = \cos5\theta$

$a^2 - b^2 = \cos^2 5\theta - i^2 \sin^2 5\theta = \cos^2 5\theta + \sin^2 5\theta = 1$

$a^2 - d^2 = \cos^2 5\theta - \cos^2 5\theta = 0$

$a^2 + b^2 = \cos^2 5\theta - \sin^2 5\theta = \cos 10\theta = \cos\frac{5\pi}{12} \in (0,1)$

$a^2 - c^2 = \cos^2 5\theta - i^2 \sin^2 5\theta = \cos^2 5\theta + \sin^2 5\theta = 1$

10. If $\left(a+\sqrt{2}b\cos x\right)\left(a-\sqrt{2}b\cos y\right) = a^2 - b^2$, where a > b > 0, then $\frac{dx}{dy}$ at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ is :

(1) $\frac{a-2b}{a+2b}$ (2) $\frac{2a+b}{2a-b}$ (3) $\frac{a-b}{a+b}$ (4) $\frac{a+b}{a-b}$

Ans (4)

Sol. $\left(a+\sqrt{2}b\cos x\right)\left(a-\sqrt{2}b\cos y\right) = a^2 - b^2$

Differentiate w.r.t. 'x'

$\left(0-\sqrt{2}b\sin x\right)\left(a-\sqrt{2}b\cos y\right) + \left(a+\sqrt{2}b\cos x\right).\left(0+\sqrt{2}b\sin y\right)y' = 0$

At $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

$(-b)(a-b) + (a+b)by' = 0$

$y' = \frac{a-b}{a+b}$

$\frac{dy}{dx} = \frac{a-b}{a+b}$ $\quad \boxed{\therefore \frac{dx}{dy} = \frac{a+b}{a-b}}$

11. A survey shows that 63% of the people in a city read newspaper A whereas 76% read newspaper B. If x% of the people read both the newspapers, then a possible value of x can be :

(1) 65 (2) 29 (3) 37 (4) 55

Ans (4)

Sol. A = Person who read newspaper of type A

B = Person who read newspaper of type B

n(A) = 63%

n(B) = 76%

$n(A \cap B) = x\%$

Let $n(U) = 100$

$n(A) = 63, n(B) = 76, n(A \cap B) = x$

$n(A \cup B) = n(A) + n(B) - n(A \cap B) \le 100$

$63 + 76 - x \le 100$

$x \ge 39$

$n(A \cap B) \le \min\{n(A), n(B)\}$

$\therefore 39 \le x \le 63$

12. Let [t] denote the greatest integer $\le t$. Then the equation in x, $[x]^2 + 2[x + 2] - 7 = 0$ has :

(1) no integral solution

(2) infinitely many solutions

(3) exactly two solutions

(4) exactly four integral solutions

Ans (2)

Sol. We have $[x]^2 + 2[x + 2] - 7 = 0$

$[x]^2 + 2([x] + 2) - 7 = 0$ $\quad$ $[[x + I] = [x] + I$

Let $[x] = t$ $\quad$ $I \in$ integer]

$\Rightarrow t^2 + 2(t + 2) - 7 = 0$

$\Rightarrow t^2 + 2t - 3 = 0$

$(t + 3)(t - 1) = 0$

$t = -3$ or $t = 1$

$[x] = -3$ or $[x] = 1$

$-3 \le x < -2$ or $1 \le x < 2$

$\therefore x \in [-3, -2) \cup [1, 2)$

Hence, infinitely many solutions.

13. Let f be a twice differentiable function on (1, 6). If f(2) = 8, $f'(2) = 5, f'(x) \ge 1$ and $f''(x) \ge 4$, for all $x \in (1, 6)$, then :

(1) $f(5) + f'(5) \ge 28$

(2) $f(5) + f'(5) \le 26$

(3) $f(5) \le 10$

(4) $f'(5) + f''(5) \le 20$

Ans (1)

Sol. We have

$f'(x) \ge 1$

$\int_2^5 f'(x)\,dx \ge \int_2^5 1\,dx$

$\Rightarrow [f(x)]_2^5 \ge (5-2)$

$\Rightarrow f(5) - f(2) \ge 3$

$\Rightarrow f(5) \ge 3 + f(2)$

$f(5) \ge 11$ ....(I)

$f''(x) \ge 4$

$\int_2^5 f''(x)\,dx \ge \int_2^5 4\,dx$

$\Rightarrow [f'(x)]_2^5 \ge 4(5-2)$

$\Rightarrow f'(5) - f'(2) \ge 12$

$\Rightarrow f'(5) \ge 12 + f'(2) = 17$

$\therefore f'(5) \ge 17$ ....(II)

using (I) and (II)

$f(5) + f'(5) \ge 28$

14. Let $u = \frac{2z+i}{z-ki}, z = x+iy$ and $k > 0$. If the curve represented by Re(u) + Im(u) = 1 intersects the y–axis at the points P and Q where PQ = 5, then the value of k is :

(1) 3/2 (2) 2 (3) 4 (4) 1/2

Ans (2)

Sol. $u = \frac{2z+i}{z-ki}$ $z = x + iy, k > 0$

Put z = x + iy in the above relation

$$u = \frac{2x+2iy+i}{x+iy-ki}$$

$$u = \frac{2x+(2y+1)i}{x+(y-k)i} \times \frac{x-(y-k)i}{x-(y-k)i}$$

$$u = \frac{2x^2 - 2x(y-k)i + x(2y+1)i + (2y+1)(y-k)}{x^2+(y-k)^2}$$

$$Re(u) = \frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2}$$

$$Im(u) = \frac{(2y+1)x - 2x(y-k)}{x^2+(y-k)^2} = \frac{x(1+2k)}{x^2+(y-k)^2}$$

Let Curve C : Re(u) + Im(u) = 1

$$\Rightarrow \frac{2x^2+(2y+1)(y-k)}{x^2+(y-k)^2} + \frac{x(1+2k)}{x^2+(y-k)^2} = 1$$

$\Rightarrow 2x^2 + (2y + 1)(y - k) + x(1 + 2k) = x^2 + (y - k)^2$ .... (1)

Curve cuts y–axis at two points P and Q

For point of intersection at y–axis

Put x = 0 in the equation (1)

$0 + (2y + 1)(y - k) + 0 = 0 + (y - k)^2$

$\Rightarrow 2y^2 - 2yk + y - k = y^2 - 2yk + k^2$

$\Rightarrow y^2 + y - k - k^2 = 0$ (roots y_1, y_2)

$\Rightarrow PQ = |y_1 - y_2| = \left|\frac{\sqrt{D}}{a}\right|$

$\Rightarrow 5 = \left|\frac{\sqrt{1+4k+4k^2}}{1}\right|$

$\Rightarrow 25 = 1 + 4k + 4k^2$

$\Rightarrow 4(k^2 + k) = 24$

$\Rightarrow k^2 + k - 6 = 0$

$\Rightarrow (k + 3)(k - 2) = 0$

$\therefore k = -3$ or $k = 2$

Since $k > 0$

$\therefore k = 2$ (accepted)

15. If $1 + (1 - 2^2.1) + (1 - 4^2.3) + (1 - 6^2.5) + ... + (1 - 20^2.19) = \alpha - 220\beta$, then an ordered pair (α, β) is equal to :

(1) (10, 97) (2) (11, 97) (3) (11, 103) (4) (10, 103)

Ans (3)

Sol. $\text{Sum} = 1 + \sum_{r=1}^{10}(1-(2r-1)(2r)^2)$

$= 1 + \sum_{r=1}^{10}(1 - 8r^3 + 4r^2)$

$= 1 + 10 - 8\sum_{r=1}^{10} r^3 + 4\sum_{r=1}^{10} r^2$

$= 11 - 8\left(\frac{10\times 11}{2}\right)^2 + 4\left(\frac{10\times 11\times 21}{6}\right)$

$= 11 - 8(55)^2 + 55 \times 28$

$= 11 + 55(28 - 440) = 11 + 55 \times (-412)$

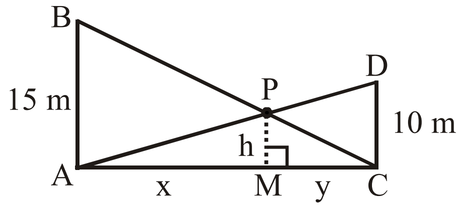
$= 11 - 220 \times 103 \quad \therefore \alpha = 11, \beta = 103$

16. Two vertical poles AB = 15 m and CD = 10 m are standing apart on a horizontal ground with points A and C on the ground. If P is the point of intersection of BC and AD, then the height of P (in m) above the line AC is :

(1) 10/3 (2) 5 (3) 6 (4) 20/3

Ans (3)

Sol.



Let AM = x, CM = y

In ΔACD

$$\frac{h}{x}=\frac{10}{x+y} \quad .... (1)$$

In ΔABC

$$\frac{h}{y}=\frac{15}{x+y} \quad .... (2)$$

(1) $\div$ (2)

$$\frac{y}{x}=\frac{10}{15}$$

$$\therefore \frac{y}{x}=\frac{2}{3} \quad .... (3)$$

Using (1) & (3)

$$\frac{h}{x}=\frac{10}{x\left(1+\frac{y}{x}\right)}$$

$$\Rightarrow h=\frac{10}{1+\frac{2}{3}}$$

h = 6

17. The mean and variance of 8 observations are 10 and 13.5, respectively. If 6 of these observations are 5, 7, 10, 12, 14, 15, then the absolute difference of the remaining two observations is :

(1) 5 (2) 9 (3) 3 (4) 7

Ans (4)

Sol. Let remaining two observations are a and b.

$$\text{Mean} = 10 = \frac{5+7+12+10+15+14+a+b}{8}$$

$\Rightarrow 80 = 63 + a + b$

$a + b = 17$ ....(I)

$$\text{Variance} = \sigma^2 = \frac{1}{n}\sum x_i^2 - \left(\bar{x}\right)^2$$

$$\Rightarrow 13.5 = \frac{1}{8}\,(25 + 49 + 144 + 100 + 225 + 196 + a^2 + b^2) - 100$$

$\Rightarrow 108 = 739 + a^2 + b^2 - 800$

$\Rightarrow 908 = 739 + a^2 + b^2$

$\Rightarrow a^2 + b^2 = 169$

$\Rightarrow (a + b)^2 - 2ab = 169$

$\Rightarrow 289 - 169 = 2ab$ [use (I)]

$ab = 60$

$(a - b)^2 = (a + b)^2 - 4ab$

$\Rightarrow (a - b)^2 = 289 - 240$

$\Rightarrow (a - b)^2 = 49$

$\therefore |a - b| = 7$

18. Let $f(x) = \int \frac{\sqrt{x}}{(1+x)^2}dx\,(x \geq 0)$. then

f(3) – f(1) is equal to

(1) $\frac{\pi}{6}+\frac{1}{2}-\frac{\sqrt{3}}{4}$ (2) $\frac{\pi}{12}+\frac{1}{2}-\frac{\sqrt{3}}{4}$ (3) $-\frac{\pi}{6}+\frac{1}{2}+\frac{\sqrt{3}}{4}$ (4) $-\frac{\pi}{12}+\frac{1}{2}+\frac{\sqrt{3}}{4}$

Ans (2)

Sol. Let $\sqrt{x} = \tan\theta$

$x = \tan^2\theta$

$\Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$

$$I = \int \frac{\tan\theta.2\tan\theta\sec^2\theta\, d\theta}{(1+\tan^2\theta)^2}$$

$= \int \frac{2\tan^2\theta \sec^2\theta}{\sec^4\theta} d\theta$

$= \int \frac{2(\sec^2\theta - 1}{\sec^2\theta} d\theta$

$= \int 2(1-\cos^2\theta)\, d\theta$

$= \int 2\sin^2\theta\, d\theta$

$= \int (1-\cos 2\theta)\, d\theta$

$= \theta - \frac{\sin 2\theta}{2} + C$

$= \theta - \frac{\tan\theta}{1+\tan^2\theta} + C$

$f(x) = \tan^{-1}\left(\sqrt{x}\right) - \frac{\sqrt{x}}{1+x} + C$

$f(3) = \tan^{-1}\left(\sqrt{3}\right) - \frac{\sqrt{3}}{4} + C$

$f(1) = \tan^{-1}(1) - \frac{1}{2} + C$

$f(3) - f(1) = \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1) + \frac{1}{2} - \frac{\sqrt{3}}{4}$

$= \frac{\pi}{3} - \frac{\pi}{4} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

$= \frac{\pi}{12} + \frac{1}{2} - \frac{\sqrt{3}}{4}$

19. The integral $\int \left(\frac{x}{x\sin x + \cos x}\right)^2 dx$ is equal to (where C is a constant of integration) :

(1) $\tan x + \frac{x\sec x}{x\sin x + \cos x} + C$

(2) $\tan x - \frac{x\sec x}{x\sin x + \cos x} + C$

(3) $\sec x - \frac{x\tan x}{x\sin x + \cos x} + C$

(4) $\sec x + \frac{x\tan x}{x\sin x + \cos x} + C$

Ans (2)

Sol. $\int \frac{x^2}{(x\sin x + \cos x)^2}dx$

$= \int (x\sec x).\frac{(x\cos x)}{(x\sin x + \cos x)^2}dx$

$= x\sec x \int \frac{x\cos x}{(x\sin x + \cos x)^2}dx - \int (\sec x + x\sec x\tan x)\left(\int \frac{x\cos x}{(x\sin x + \cos x)^2}dx\right)dx$

$\int \frac{x\cos x}{(x\sin x + \cos x)^2}dx = -\frac{1}{x\sin x + \cos x}$

[Use substitution t = xsinx + cosx
dt = xcosxdx]

$= x\sec x.\left(\frac{-1}{x\sin x + \cos x}\right) - \int \sec^2 x(\cos x + x\sin x)\left(\frac{-1}{x\sin x + \cos x}\right)dx$

$= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec^2 x dx$

$= \tan x - \frac{x\sec x}{x\sin x + \cos x} + c$

20. Let α and β be the roots of $x^2 - 3x + p = 0$ and γ and δ be the roots of $x^2 - 6x + q = 0$. If $\alpha, \beta, \gamma, \delta$ from a geometric progression. Then ratio $(2q + p) : (2q - p)$ is :

(1) 3 : 1 (2) 33 : 31 (3) 9 : 7 (4) 5 : 3

Ans (3)

Sol. $x^2 - 3x + p = 0 \langle^{\alpha}_{\beta}$

$\alpha + \beta = 3; \alpha\beta = p$

$x^2 - 6x + q = 0 \langle^{\gamma}_{\delta}$

$\gamma + \delta = 6; \gamma\delta = q$

$\because$ $\alpha, \beta, \gamma, \delta$ are in G.P.

Let α, $\beta = \alpha r$, $r = \alpha r^2$, $\delta = \alpha r^3$, where r is common ratio of the G.P.

$\alpha + \alpha r = 3$ ....(I)

$\alpha r^2 + \alpha r^3 = 6$ ....(II)

(I) ÷ (II)

$$\frac{\alpha(1+r)}{\alpha r^2(1+r)} = \frac{1}{2}$$

$\therefore\ r^2 = 2$

$$\frac{2q+p}{2q-p} = \frac{(2\gamma\delta+\alpha\beta)}{(2\gamma\delta-\alpha\beta)} = \frac{(2\alpha^2r^5+\alpha^2r)}{(2\alpha^2r^5-\alpha^2r)}$$

$$= \frac{2r^4+1}{2r^4-1} = \frac{2\times4+1}{2\times4-1} = \frac{9}{7}$$

21. If the system of equations

$x - 2y + 3z = 9$

$2x + y + z = b$

$x - 7y + az = 24$, has infinitely many solutions, then a – b is equal to __________.

Ans (5)

S. $x - 2y + 3z = 9$

$2x + y + z = b$

$x - 7y + az = 24$

Here, for infinite solution

$D = D_x = D_y = D_z = 0$

$$D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ 1 & -7 & a \end{vmatrix} = 5a - 40 = 5(a-8)$$

$\therefore\ D = 0$ $\quad\therefore\ a = 8$

$$D_x = \begin{vmatrix} 9 & -2 & 3 \\ b & 1 & 1 \\ 24 & -7 & 8 \end{vmatrix} = 15 - 5b = (3-b)5$$

$\therefore\ D_x = 0$ $\quad\therefore\ b = 3$

$$D_y = \begin{vmatrix} 1 & 9 & 3 \\ 2 & b & 1 \\ 1 & 24 & 8 \end{vmatrix} = 5b - 15$$

$\therefore D_y = 0$ $\qquad\qquad \therefore b = 3$

$$D_z = \begin{vmatrix} 1 & -2 & 9 \\ 2 & 1 & b \\ 1 & -7 & 24 \end{vmatrix} = 5b - 15$$

$\therefore D_z = 0$ $\qquad\qquad \therefore b = 3$

$\therefore$ When a = 8 and b = 3

$D = D_x = D_y = D_z = 0$

$\therefore$ System has infinite solutions

$\therefore$ a – b = 8 – 3 = 5

22. The probability of a man hitting a target is $\frac{1}{10}$. The least number of shots required, so that the probability of his hitting the target at least once is greater than $\frac{1}{4}$, is

Ans 3

Sol. Let number of shot required = n

Prob (To hit target at least once)

= 1 – P (not hitting in n attempt)

H = an event of hitting the target

$P(H) = \frac{1}{10}, \quad P(\overline{H}) = \frac{9}{10}$

$= 1 - \left(\frac{9}{10}\right)^n$

$1 - \left(\frac{9}{10}\right)^n > \frac{1}{4}$

$\frac{3}{4} > \left(\frac{9}{10}\right)^n$

$n = 1 \quad \frac{3}{4} > \frac{9}{10} \quad \times$

$n = 2 \quad \frac{3}{4} > \frac{81}{100} \quad \times$

$n = 3 \quad \frac{3}{4} > \frac{729}{1000} \quad \surd$

Minimum number of shots = 3.

23. If the equation of a plane P, passing through the intersection of the planes, x + 4y – z + 7 = 0 and 3x + y + 5z = 8 is ax + by + 6z = 15 for some $a, b \in \mathbf{R}$, then the distance of the point (3, 2, –1) from the plane P is ______________.

Ans 3

S. Let the required plane P be

$x + 4y - z + 7 + \lambda(3x + y + 5z - 8) = 0$

$\Rightarrow x(1 + 3\lambda) + y(4 + \lambda) + z(5\lambda - 1) + 7 - 8\lambda = 0$

$\Rightarrow \frac{x}{\frac{8\lambda-7}{1+3\lambda}} + \frac{y}{\frac{8\lambda-7}{4+\lambda}} + \frac{z}{\frac{8\lambda-7}{5\lambda-1}} = 1$ .... (1)

Compare with the given equation of the plane P ax + by + 6z = 15 that is

$\frac{x}{\frac{15}{a}} + \frac{y}{\frac{15}{b}} + \frac{z}{\frac{15}{6}} = 1$ .... (2)

Since (1) & (2) represent same plane

$\therefore \frac{8\lambda - 7}{5\lambda - 1} = \frac{15}{6}$

$\therefore \lambda = -1$

$\therefore$ Equation of required plane P is

$-2x + 3y - 6z + 15 = 0$

$\Rightarrow 2x - 3y + 6z - 15 = 0$

Distance of the point (3, 2, –1) from the plane P $= \frac{|2.3 - 3.2 - 6 - 15|}{\sqrt{4 + 9 + 36}} = \frac{21}{\sqrt{49}} = 3$

24. Let $\left(2x^2+3x+4\right)^{10}=\sum_{r=0}^{20} a_r x^r$. Then $\frac{a_7}{a_{13}}$ is equal to __________.

Ans 8

Sol. $(2x^2 + 3x + 4)^{10} = \sum_{r=0}^{20} a_r x^r$

General term of $(2x^2 + 3x + 4)^{10}$

$=\frac{10!}{p_1!p_2!p_3!}\left(2x^2\right)^{p_1}\left(3x\right)^{p_2}\left(4\right)^{p_3}$

Where $p_1 + p_2 + p_3 = 10$

and $0 \le p_i \le 10; p_i \in W, i = 1,2,3$

a_7 = coefficient of x^7

a_{13} = coefficient of x^{13}.

$=\frac{10!}{p_1!p_2!p_3!}2^{p_1}.3^{p_2}.4^{p_3}\,x^{2p_1+p_2}$

For a_7 : $2p_1 + p_2 = 7$

p_1	p_2	p_3	Coefficient of x^7
3	1	6	$\frac{10!}{3!.1!.6!}2^3.3^1.4^6=\frac{10!}{3!.1!.6!}2^{15}.3^1$
2	3	5	$\frac{10!}{2!.3!.5!}2^2.3^3.4^5=\frac{10!}{2!.3!.5!}2^{12}.3^3$
1	5	4	$\frac{10!}{1!.5!.4!}2^1.3^5.4^4=\frac{10!}{1!.5!.4!}2^9.3^5$
0	7	3	$\frac{10!}{7!.3!}1.3^7.4^3=\frac{10!}{7!.3!}2^6.3^7$

$a^7=2^3\left(\frac{10!}{3!.1!.6!}2^{12}.3^1+\frac{10!}{2!.3!.5!}2^9.3^3+\frac{10!}{1!.5!.4!}2^6.3^5+\frac{10!}{7!.3!}2^3.3^7\right)$

For a_3 : $2p_1 + p_2 = 13$

p_1	p_2	p_3	Coefficient of x^{13}
6	1	3	$\frac{10!}{6!.1!.3!}2^6.3^1.4^3 = \frac{10!}{6!.1!.3!}2^{12}.3^1$
5	3	2	$\frac{10!}{5!.3!.2!}2^5.3^3.4^2 = \frac{10!}{5!.3!.2!}2^9.3^3$
4	5	1	$\frac{10!}{4!.5!.1!}2^4.3^5.4^1 = \frac{10!}{4!.5!.1!}2^6.3^5$
3	7	0	$\frac{10!}{3!.7!}2^3.3^7 = \frac{10!}{3!.7!}2^3.3^7$

$$a^{13} = \left(\frac{10!}{6!.1!.3!}2^{12}.3^1 + \frac{10!}{5!.3!.2!}2^9.3^3 + \frac{10!}{4!.5!.1!}2^6.3^5 + \frac{10!}{3!.7!}2^3.3^7\right)$$

$$\frac{a_7}{a_{13}} = 2^3 = 8$$

25. Suppose a differentiable function f(x) satisfies the identity $f(x + y) = f(x) + f(y) + xy^2 + x^2y$, for all real x and y. If $\lim_{x\to 0}\frac{f(x)}{x} = 1$, then $f'(3)$ is equal to ____________.

Ans 10

Sol.
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+x^2h+h^2x-f(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(h)}{h}+x^2+hx\right]$$

$f'(x) = 1 + x^2$

$\therefore f'(3) = 1 + 9 = 10$