JEE Main September 2020

Question Paper With Text Solution

3 September| Shift-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE MAIN SEP 2020 | 3 SEP SHIFT-2

1. $\int_0^{1/2} \frac{x^2}{(1-x^2)^{\frac{3}{2}}}dx$ is $\frac{k}{6}$, then k is equal to :

(1) $2\sqrt{3}-\pi$ (2) $2\sqrt{3}+\pi$ (3) $3\sqrt{2}+\pi$ (4) $3\sqrt{2}-\pi$

Ans. (1)

Sol. $\frac{k}{6} = \int_0^{\frac{1}{2}} \frac{x^2}{(1-x)^{3/2}}dx$ $\qquad x = \sin\theta; dx = \cos\theta\, d\theta$

$\Rightarrow \frac{k}{6} = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{(1-\sin^2\theta)^{3/2}}.\cos\theta d\theta \Rightarrow \frac{k}{6} = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos^3\theta}.\cos\theta d\theta$

$\Rightarrow \frac{k}{6} = \int_0^{\frac{\pi}{6}} \tan^2\theta d\theta = \int_0^{\frac{\pi}{6}} (\sec^2\theta - 1)\, d\theta \Rightarrow \frac{k}{6} = (\tan\theta - \theta)_0^{\pi/6} = \left(\frac{1}{\sqrt{3}} - \frac{\pi}{6}\right) = \frac{2\sqrt{3}-\pi}{6}$

$\Rightarrow k = 2\sqrt{3}-\pi$

2. Let A be a 3 × 3 matrix such that adj $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & -1 \end{bmatrix}$ and B = adj(adj A).

If $|A| = \lambda$ and $|(B^{-1})^T| = \mu$, then the ordered pair, $(|\lambda|, \mu)$ is equal to :

(1) (3, 81) (2) $\left(9, \frac{1}{9}\right)$ (3) $\left(3, \frac{1}{81}\right)$ (4) $\left(9, \frac{1}{81}\right)$

Ans. (3)

Sol. $|\text{adj } A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & -2 & -1 \end{vmatrix} = |A|^{n-1}$

$\therefore |A|^2 = 9$

$|A| = \pm 3$

$|A| = \lambda = \pm 3$

$|\lambda| = 3$

$B = \text{adj (adj } A)$

$|B| = |\text{adj (adj } A)|$

$|B| = |A|^{(n-1)^2}$

$|B| = |A|^4$

$|B| = 81$

$\left|\left(B^{-1}\right)^T\right| = \mu$

$\left|B^{-1}\right| = \mu$

$\dfrac{1}{|B|} = \mu$

$\mu = \dfrac{1}{81}$

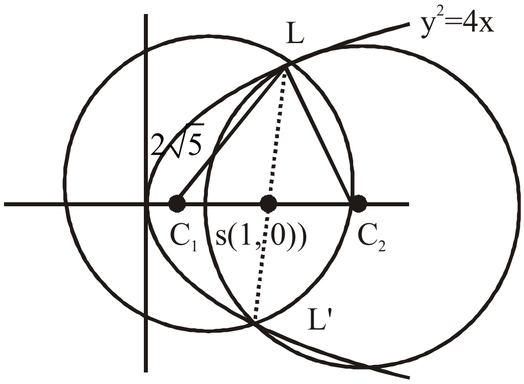
$= \left(3, \dfrac{1}{81}\right)$

3. Let the latus ractum of the parabola $y^2 = 4x$ be the common chord to the circels C_1 and C_2 each of them having radius $2\sqrt{5}$. Then, the distance betwen the centres of the circles C_1 and C_2 is :

(1) 12 (2) $8\sqrt{5}$ (3) 8 (4) $4\sqrt{5}$

Ans. (3)

Sol.



$C_1 C_2 = 2C_1 S = 2\sqrt{20-4} = 8$

4. Let p, q, r be three statements such that the truth value of $(p \wedge q) \to (\sim q \vee r)$ is F. Then the truth value of p, q, r are respectively :

(1) T, T, T (2) T, F, T (3) F, T, F (4)T,T,F

Ans. (4)

Sol. $(p \wedge q) \to (\sim q \vee r)$

$A \to B =$ false

A should be true and B should be false

$p \wedge q \to$ True | $\sim q \vee r \to$ False

$q \Rightarrow$ True | $\sim q \Rightarrow$ False

$p \Rightarrow$ True | $r \Rightarrow$ False

hence | $q \Rightarrow$ True

$p \Rightarrow$ True

$q \Rightarrow$ True

$r \Rightarrow$ false

5. If z_1, z_2 are complex numbers such that Re $(z_1) = |z_1 - 1|$, Re $(z_2) = |z_2 - 1|$ and arg $(z_1 - z_2) = \frac{\pi}{6}$, then $\text{Im}(z_1 + z_2)$ is equal to :

(1) $\frac{\sqrt{3}}{2}$ (2) $\frac{1}{\sqrt{3}}$ (3) $2\sqrt{3}$ (4) $\frac{2}{\sqrt{3}}$

Ans. (3)

Sol. Let $z_1 = x_1 + iy_1$

$z_2 = x_2 + 1y_2$

Re $(z_1) = |z_1 - 1|$

$x_1 = |x_1 + iy_1 - 1|$

$x_1^2 = (x-1)^2 + y_1^2$

$y_1^2 + 1 = 2x_1$ ....(1)

Re $(z_2) = |z_2 - 1|$

$y_2^2 + 1 = 2x_2$ ....(2)

By (1) – (2)

$$y_1^2 - y_2^2 = 2(x_1 - x_2)$$

$$(y_1 + y_2)(y_1 - y_2) = 2(x_1 - x_2)$$

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{2}{y_1 + y_2} \quad ....(3)$$

$$\arg(z_1 - z_2) = \frac{\pi}{6}$$

$$\tan^{-1}\left(\frac{y_1 - y_2}{x_1 - x_2}\right) = \frac{\pi}{6}$$

$$\frac{y_1 - y_2}{x_1 - x_2} = \frac{1}{\sqrt{3}}$$

by (3)

$$\frac{1}{\sqrt{3}} = \frac{2}{y_1 + y_2}$$

$$y_1 + y_2 = 2\sqrt{3}$$

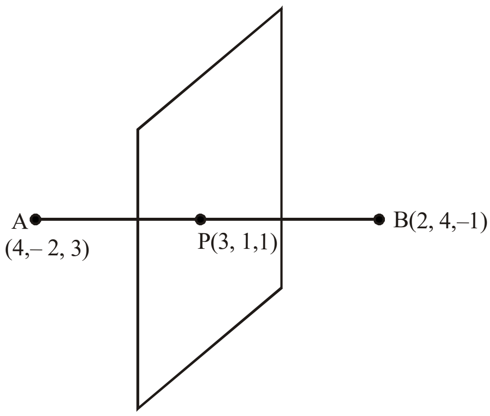
$$\text{Img}(z_1 + z_2) = 2\sqrt{3}$$

6. The plane which bisects the line joining the points (4, –2, 3) and (2, 4, –1) at right angles also passes through the point :

(1) (0, –1,1) (2) (4, 0, –1) (3) (0, 1, –1) (4) (4, 0, 1)

Ans. (2)

Sol.



$\vec{n} = 2\hat{i} - 6\hat{j} + 4\hat{k}$ (Normal of plane)

$\vec{a} = 3\hat{i} + \hat{j} + \hat{k}$ (Point on plane)

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$(x\hat{i} + y\hat{j} + z\hat{k}).(2\hat{i} - 6\hat{j} + 4\hat{k}) = 6 - 6 + 4$

$2x - 6y + 4z = 4$

It passes through $(4, 0, -1)$

7. Let R_1 and R_2 be two relations defined as follows :

$R_1 = \{(a, b) \in R^2 : a^2 + b^2 \in Q\}$ and

$R_2 = \{(a, b) \in R^2 : a^2 + b^2 \notin Q\}$, where Q is the set of all rational numbers. Then :

(1) R_1 and R_2 are both transitive
(2) R_2 is transitive but R_1 is not transitive
(3) Neither R_1 nor R_2 is transitive
(4) R_1 is transitive but R_2 is not transitive

An.s. (3)

Sol. For R_1 let $a = \left(5+\sqrt{3}\right)^{\frac{1}{2}}, b = \left(5-\sqrt{3}\right)^{\frac{1}{2}}, c = \left(4+\sqrt{3}\right)^{\frac{1}{2}}$

$aR_1b \Rightarrow a^2 + b^2 = (5+\sqrt{3}) + (5-\sqrt{3}) = 10 \in Q$

$aR_1c \Rightarrow b^2 + c^2 = (5-\sqrt{3}) + (4+\sqrt{3}) = 9 \in Q$

$aR_1c \Rightarrow a^2 + c^2 = (5+\sqrt{3}) + (4+\sqrt{3}) = 9 + 2\sqrt{36} \notin Q$

$\therefore R_1$ is not transitive

For R_2 let $a = \left(5+\sqrt{3}\right)^{\frac{1}{2}}, b = 2, c = \left(5-\sqrt{3}\right)^{\frac{1}{2}}$

$aR_1c \Rightarrow a^2 + b^2 = (5+\sqrt{3}) + (4) = 9 + \sqrt{3} \notin Q$

$bR_2b \Rightarrow b^2 + c^2 = (4) + (5-\sqrt{3}) = 9 - \sqrt{3} \notin Q$

$aR_2c \Rightarrow a^2 + c^2 = (5+\sqrt{3}) + (5-\sqrt{3}) = 10 \in Q$

$\therefore R_2$ is not transitive

8. $\lim\limits_{x\to a}\dfrac{(a+2x)^{\frac{1}{3}}-(3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}}-(4x)^{\frac{1}{3}}}\ (a\neq 0)$ is equal to:

(1) $\left(\dfrac{2}{9}\right)^{\frac{4}{3}}$ (2) $\left(\dfrac{2}{9}\right)\left(\dfrac{2}{3}\right)^{\frac{1}{3}}$ (3) $\left(\dfrac{2}{3}\right)\left(\dfrac{2}{9}\right)^{\frac{1}{3}}$ (4) $\left(\dfrac{2}{3}\right)^{\frac{4}{3}}$

Ans. 2

Sol. $\lim\limits_{x\to a}\dfrac{(a+2x)^{\frac{1}{3}}-(3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}}-(4x)^{\frac{1}{3}}}$

$$a-b=\frac{a^3-b^3}{a^2+ab+b^2}$$

$$\lim_{x\to a}\frac{(a+2x-3x)}{(a+2x)^{\frac{2}{3}}+(a+2x)^{\frac{1}{3}}(3x)^{\frac{1}{3}}+(3x)^{\frac{2}{3}}}\times\frac{(3a+x)^{\frac{2}{3}}+(3a+x)^{\frac{1}{3}}(4x)^{\frac{1}{3}}+(4x)^{\frac{2}{3}}}{(3a+x-4x)}$$

$$\lim_{x\to a}\frac{(a-x)}{3(a-x)}\times\frac{\left((4a)^{\frac{2}{3}}+(4a)^{\frac{1}{3}}(4a)^{\frac{1}{3}}+(4a)^{\frac{2}{3}}\right)}{\left((3a)^{\frac{2}{3}}+(3a)^{\frac{1}{3}}(3a)^{\frac{1}{3}}+(3a)^{\frac{2}{3}}\right)}$$

$$\lim_{x\to a}\frac{1}{3}\times\left(\frac{3(4a)^{\frac{2}{3}}}{3(3a)^{\frac{2}{3}}}\right)$$

$$\frac{1}{3}\times\left(\frac{4}{3}\right)^{\frac{2}{3}}$$

$$\frac{2}{3}\times\left(\frac{2}{9}\right)^{\frac{1}{3}}$$

9. Let a, b, c $\in$ R be such that $a^2+b^2+c^2=1$. If $a\cos\theta=b\cos\left(\theta+\dfrac{2\pi}{3}\right)=c\cos\left(\theta+\dfrac{4\pi}{3}\right)$, where $\theta=\dfrac{\pi}{9}$, then the angle between the vectors $a\hat{i}+b\hat{j}+c\hat{k}$ and $b\hat{i}+c\hat{j}+a\hat{k}$ is :

(1) $\dfrac{\pi}{2}$ (2) $\dfrac{\pi}{9}$ (3) $\dfrac{2\pi}{3}$ (4) 0

Ans. (1)

Sol. $a\cos\theta = b\cos\left(\theta+\frac{2\pi}{3}\right) = c\ \cos\left(\theta+\frac{4\pi}{3}\right) = K$

$\frac{K}{a} = \cos\theta$

$\frac{K}{b} = \cos\left(\frac{2\pi}{3}+0\right)$

$\frac{K}{c} = \cos\left(\frac{4\pi}{3}+\theta\right)$

$\frac{K}{a}+\frac{K}{b}+\frac{K}{c} = \cos\theta+\left(\cos\frac{2\pi}{3}+\theta\right)+\cos\left(\frac{4\pi}{3}+\theta\right)$

$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 0$

$ab + bc + ca = 0$

$\cos\alpha = \frac{ab+bc+ca}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2}}$

$\cos\alpha = 0$

$= \frac{\pi}{2}$

10. If the term independent of x in the expansion of $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ is k, then 18 k is equal to :

(1) 9 (2) 11 (3) 5 (4) 7

Ans. (4)

Sol. Let T_{r+1} be term independent of x.

$T_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r$

$T_{r+1} = {}^9C_r\left(\frac{3}{2}\right)^{9-r}\left(-\frac{1}{3}\right)^r x^{18-2r-r}$

$18 - 3r = 0$

$r = 6$

$$T_{6+1} = {}^9C_6\left(\frac{3}{2}\right)^3\left(\frac{-1}{3}\right)^6$$

$$=\frac{9.8.7}{3.2.1}\times\frac{1}{8}\times\frac{1}{27}$$

$$T_7 = \frac{7}{18} = K$$

$$18K = 18\times\frac{7}{18} = 7$$

11. Let e_1 and e_2 be the eccentricities of the ellipse, $\frac{x^2}{25}+\frac{y^2}{b^2}=1(b<5)$ and the hyperbola, $\frac{x^2}{16}-\frac{y^2}{b^2}=1$ respectively satisfying $e_1e_2 = 1$. If α and β are the distances between the foci of the ellipse and the foci of the hyperbola respectively, then the ordered pair (α, β) is equal to :

(1) $\left(\frac{20}{3},12\right)$ (2) (8, 12) (3) $\left(\frac{24}{5},10\right)$ (4) (8, 10)

Ans. (4)

Sol. $e_1 = \sqrt{1-\frac{b^2}{25}}$; $e_2 = \sqrt{1+\frac{b^2}{16}}$

$$e_1e_2 = 1$$

$$\Rightarrow (e_1e_2)^2 = 1 \Rightarrow \left(1-\frac{b^2}{25}\right)\left(1+\frac{b^2}{16}\right)=1 \Rightarrow 1+\frac{b^2}{16}-\frac{b^2}{25}-\frac{b^4}{25\times16}=1$$

$$\Rightarrow \frac{9}{16.25}b^2-\frac{b^4}{25.16}=0 \Rightarrow b^2=9$$

$$e_1 = \sqrt{1-\frac{9}{25}}=\frac{4}{5}$$

$$e_2 = \sqrt{1+\frac{9}{16}}=\frac{5}{4}$$

$$\alpha = 2(5)(e_1) = 8$$

$$\beta = 2(4)(e_2) = 10$$

$$(\alpha,\beta) = (8,10)$$

12. If $x^3dy + xy\,dx = x^2dy + 2y\,dx$; $y(2) = e$ and $x > 1$, then $y(4)$ is equal to :

(1) $\frac{\sqrt{e}}{2}$ (2) $\frac{1}{2}+\sqrt{e}$ (3) $\frac{3}{2}\sqrt{e}$ (4) $\frac{3}{2}+\sqrt{e}$

Ans. (3)

Sol. $x^3dy + xydx = 2ydx + x^2dy$

By variable seprable method

$\Rightarrow (x^3 - x^2)dy = (2 - x)ydx$

$$\Rightarrow \frac{dy}{y} = \frac{2-x}{x^2(x-1)}dx$$

$$\Rightarrow \int\frac{dy}{y} = \int\frac{2-x}{x^2(x-1)}dx$$

By partial fraction

Let $\frac{2-x}{x^2(x-1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}$

$\Rightarrow 2 - x = Ax(x - 1) + B(x - 1) + Cx^2$

By comparison

$\Rightarrow C = 1, B = -2$ and $A = -1$

$$\Rightarrow \int\frac{dy}{y} = \int\left\{\frac{-1}{x}-\frac{2}{x^2}+\frac{1}{x-1}\right\}dx$$

$$\Rightarrow l\,n\,y = -\ln x + \frac{2}{x} + ln|x-1| + C$$

$\therefore y(2) = e$

$$\Rightarrow lne = -ln2 + \frac{2}{2} + ln1 + C$$

$\Rightarrow C = ln2$

$$\Rightarrow l\,n\,y = -l\,n\,x + \frac{2}{x} + ln|x-1| + ln2$$

at $x = 4$

$\Rightarrow l\,n\,y(4) = -l\,n\,4 + \frac{1}{2} + ln3 + ln2$

$\Rightarrow l\,n\,y(4) = l\,n\left(\frac{3}{2}\right) + \frac{1}{2} = ln\left(\frac{3}{2}e^{\frac{1}{2}}\right)$

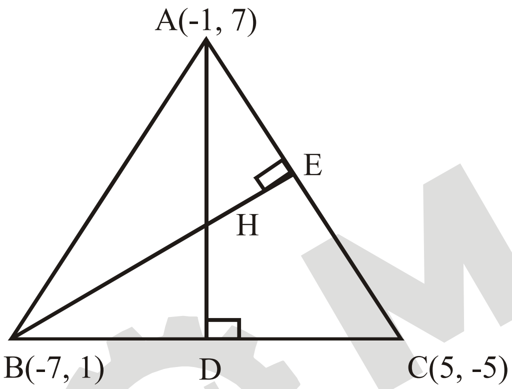
$\Rightarrow y(4) = \frac{3}{2}e^{\frac{1}{2}}$

13. If a ΔABC has vertices A(–1, 7), B(–7, 1) and C(5, –5), then its orthocentre has coordinates :

(1) $\left(-\frac{3}{5}, \frac{3}{5}\right)$ (2) (–3, 3) (3) $\left(\frac{3}{5}, -\frac{3}{5}\right)$ (4) (3, –3)

Ans. (2)

Sol.



$m_{BC} = \frac{6}{-12} = -\frac{1}{2}$

$\therefore$ Equation of AD is y - 7 = 2 (x + 1)

y = 2x + 9 ....(1)

$m_{AC} = \frac{12}{-6} = -2$

$\therefore$ Equation of BE is

$y - 1 = \frac{1}{2}(x+7)$ ....(2)

by (i) & (ii)

x = –3

y = 3

14. If $\int \sin^{-1}\left(\sqrt{\frac{x}{1+x}}\right)dx = A(x)\tan^{-1}\left(\sqrt{x}\right) + B(x) + C$, where C is a constant of integration, then the ordered pair (A(x), B(x)) can be :

(1) $(x+1, \sqrt{x})$ (2) $(x-1, \sqrt{x})$ (3) $(x-1, -\sqrt{x})$ (4) $(x+1, -\sqrt{x})$

Ans. (4)

Sol. $I = \int \sin^{-1}\left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)dx$

$\int \tan^{-1}\left(\underset{\text{I}}{\sqrt{x}}\right)\underset{\text{II}}{dx} = x\tan^{-1}\sqrt{x} - \int \frac{1}{1+x}.\frac{1}{2\sqrt{x}}.xdx + C = x\tan^{-1}\sqrt{x} - \frac{1}{2}\int \frac{t.2t.dt}{1+t^2} + C\left(x = t^2\right)$

$= x\tan^{-1}\sqrt{x} - \int \frac{t^2}{1+t^2}dt + C = x\tan^{-1}\sqrt{x} - t + \tan^{-1}t + C = x\tan^{-1}\sqrt{x} - \sqrt{x} + \tan^{-1}\sqrt{x} + C$

$= (x+1)\tan^{-1}\sqrt{x} - \sqrt{x} + c \quad \Rightarrow \quad A(x) = x+1 \Rightarrow B(x) = -\sqrt{x}$

15. If the sum of the series $20 + 19\frac{3}{5} + 19\frac{1}{5} + 18\frac{4}{5} + ...$ upto n^{th} term is 488 and the nth term is negative, then :

(1) n = 41 (2) n^{th} term is $-4\frac{2}{5}$ (3) n^{th} term is – 4 (4) n = 60

Ans. (3)

Sol. $S_n = 488$

$\frac{n}{2}\left(2a + (n-1)d\right) = 488$

$\frac{n}{2}\left(2\times 20 + (n-1)\left(\frac{-2}{5}\right)\right) = 488$

n = 40, 61

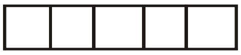
n = 40 $T_n = 20 + (40 - 1)\left(-\frac{2}{5}\right) > 0$

n = 61 $T_n = 20 + (61 - 1)\left(-\frac{2}{5}\right) < 0$ & $T_n = -4$

16. The probability that a randomly chosen 5-digit number is made from exactly two digits is :

(1) $\frac{150}{10^4}$ (2) $\frac{121}{10^4}$ (3) $\frac{134}{10^4}$ (4) $\frac{135}{10^4}$

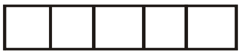
Ans. (4)

Sol. 

$9 \times 10 \times 10 \times 10 \times 10 = 9 \times 10^4$

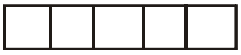
Required way

Without '0' (1,2 , ...9)



$9c_2\,(2^5 - 2) = 1080$

with '0' (1,2 , ...9)



$= 9c_1\,(1\times 2^4 - 1) = 135$

$$\text{Ans} = \frac{1080+135}{9\times 10^4}$$

$$= \frac{135}{10^4}$$

17. The set of all real values of λ for which the quadratic equations,

$(\lambda^2+1)\,x^2 - 4x + 2 = 0$ always have exactly one root in the interval (0, 1) is :

(1) (0, 2) (2) (1, 3] (3) (–3, –1) (4) (2, 4]

Ans. (2)

Sol. $f(x) = (\lambda^2 + 1)x^2 - 4\lambda x + 2$

$f(1)\, f(0) < 0$

$(\lambda^2 + 1 - 4\lambda + 2)\,(2) < 0$

$\lambda \in (1, 3)$

$\lambda = 1$

$= 2x^2 - 4x + 2 = 0$

x = 1 , 1

$\lambda = 3$

$10x^2 - 12x + 2 = 0$

$x = 1, \frac{1}{5}$

$\lambda \in (1, 3]$

18. If the surface area of a cube is increasing at a rate of 3.6 cm^2/sec, retaining its shape, thenthe rate of change of its volume (in cm^3/sec), when the length of a side of the cube is 10 cm, is :

(1) 20 (2) 10 (3) 9 (4) 18

Ans. (3)

Sol. Side of cube = l

$A = 6l^2$

$\Rightarrow \quad \frac{dA}{dt} = 12l.\frac{dl}{dt} = 3.6 \quad \Rightarrow 12(10)\frac{dl}{dt} = 3.6$

$\Rightarrow \quad \frac{dl}{dt} = 0.03$

$V = l^3 \Rightarrow \frac{dv}{dt} = 3l^2.\frac{dl}{dt} = 3(10)^2.\left(\frac{3}{100}\right) = 9$

19. Suppose f(x) is a polynomial of degree four, having critical points at –1, 0, 1. if $T = \{x \in R| f(x) = f(0)\}$, then the sum of squares of all the elements of T is :

(1) 2 (2) 6 (3) 8 (4) 4

Ans. (4)

Sol. $f'(x) = A(x+1)(x-1)x$

$f'(x) = A(x^3 - x)$

$f(x) = \frac{Ax^4}{4} - \frac{Ax^2}{2} + K$

but f(x) = f(0)

$\frac{Ax^4}{4} - \frac{Ax^2}{2} + K = K$

$A\left(2x^4 - 4x^2\right) = 0$

$x^4 - 2x^2 = 0$

$x = 0, +\sqrt{2}, -\sqrt{2}$

$0^2 + \left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2 = 4$

20. Let $x_i (1 \le i \le 10)$ be ten observation of a random variable X. If $\sum_{i=1}^{10}(x_i - p) = 3$ and $\sum_{i=1}^{10}(x_i - p)^2 = 9$ where $0 \ne p \in R$, then the standard deviation of these observation is :

(1) $\sqrt{\frac{3}{5}}$ (1) $\frac{4}{5}$ (3) $\frac{9}{10}$ (4) $\frac{7}{10}$

Ans. (3)

Sol. $\sum_{i=1}^{10}(x_i - p) = 3$

$\sum_{i=1}^{10}(x_i - p)^2 = 9$

If we add or subtract a particular number in all deta then S.D remains unchanged.

hence $\sum_{i=1}^{10} x_i = 3$

$\sum_{i=1}^{10} x^2{}_i = 9$

$\sigma^2 = \frac{1}{N}\sum_{i=1}^{10} x_i^2 - \left(\bar{x}\right)^2$

$\sigma^2 = \frac{1}{10}(9) - \left(\frac{3}{10}\right)^2$

$\sigma^2 = \frac{90-9}{100}$

$S.D. = \sigma = \frac{9}{10}$

21. Let S be the set of all integer solutions, (x, y, z) of the system of equations

$x - 2y + 5z = 0$

$-2x + 4y + z = 0$

$-7x + 14y + 9z = 0$

such that $15 \le x^2 + y^2 + z^2 \le 150$. Then, the number of elements in the set S is equal to _____.

Ans. 8

Sol. $x - 2y + 5z = 0$

$-2x + 4y + z = 0$

$-7x + 14y + 9z = 0$

$$D = \begin{vmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ -7 & 14 & 9 \end{vmatrix}$$

$D = 1(36 - 14) + 2(-18 + 7) + 5(-28 + 28)$

$D = 22 - 22 + 0 = 0$

Let $y = k$

$x - 2k + 5z = 0$

$-2x + 4k + z = 0$

$z = 0$

$x = 2k$

solutions

(x, y, z)

$(2k, k, 0)$

$s \le x^2 + y^2 + z^2 \le 150$

$s \le 4k^2 + k^2 + 0^2 \le 150$

$1 \le k^2 \le 30$

$K = \pm 5, \pm 4, \pm 3, \pm 2$

hence total number of solutions 8.

22. If m arithmetic means (A.Ms) and three geometric means (G.Ms) are inerted between 3 and 243 such that 4^{th} A.M. is equal to 2^{nd} G.M., then m is equal to ______.

Ans. 39

Sol. $3, A_1, A_2\, A_3$ ....A_m, 243

$$d = \frac{243 - 3}{m + 1} = \frac{240}{m + 1}$$

$3, G_{1,} G_2, G_3, 243$

$r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$

$r = \left(\frac{243}{3}\right)^{\frac{1}{3+1}} = (81)^{1/4} = 3$

$G_2 = 3(3)^2 = 27$

$G_2 = A_4$

$\Rightarrow 3(3)^2 = 3 + 4\left(\frac{240}{m+1}\right)$

$\Rightarrow \quad 27 = 3 + \frac{960}{m+1} \Rightarrow \quad 24(m + 1) = 4(240)$

$\Rightarrow m + 1 = 40$

$\Rightarrow m = 39$

23. If the tangent to the curve, $y = e^x$ at a point (c, e^c) and the normal to the parabola, $y^2 = 4x$ at the point $(1, 2)$ intersect at the same point on the x-axis, then the value of c is ____.

Ans. 4

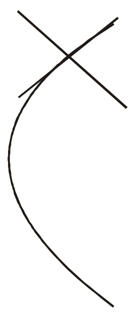
Sol. $y^2 = 4x$

$2y\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{4}{2 \times 2}$

$m_T = 1$

$m_N = -1$



$y - 2 = m_N (x - 1)$

$y - 2 = -1 (x - 1)$

$x + y - 3 = 0 \qquad .....(i)$

$y = e^x \qquad (c, e^c)$

$$\frac{dy}{dx} = e^x$$

$$\left.\frac{dy}{dx}\right|_{(c, e^c)} = e^c$$

$y - e^c = e^c\ (x - c)$ .....(ii)

$y = 0$

$x = 3$

$x = -1 + c$

$c - 1 = 3$

$c = 4$

24. The total number of 3 - digit numbers, whose sum of digits is 10, is _____.

Ans. 54

Sol. Let three digit number be ABC

$A + B + C = 10$ $A \geq 1$

Let $A = y + 1$ $B \geq 0$

$y + 1 + B + C = 10$

$y + B + C = 9$ $0 \leq y, B, C \leq 9$

Number of solutions which are non-negative

$^{9+3-1}C_{3-1}$

$= 55$

but if $y = 9$ $A = y + 1$

$= 10$

then total number of solutions $= 55 - 1 = 54$.

25. Let a plane P contain two lines

$\vec{r} = \hat{i} + \lambda(\hat{i} + \hat{j}), \lambda \in R$ and $\vec{r} = -\hat{j} + \mu(\hat{j} - \hat{k}), \mu \in R$.

If $Q(\alpha, \beta, \gamma)$ is the foot of the perpendicular drawn from the point $M(1, 0, 1)$ to P, then $3(\alpha + \beta + \gamma)$ equals _____.

Ans. 5

Sol. $\vec{p} = \hat{i} + \hat{j} + 0\hat{k}$

$\vec{q} = 0\hat{i} + \hat{j} - \hat{k}$

$$\vec{n} = \vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix}$$

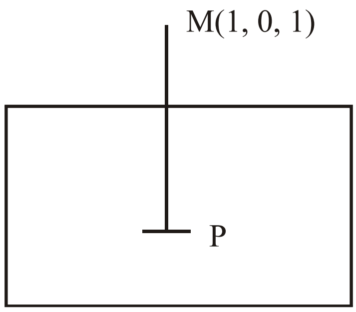
$= \hat{i}(-1-0) - \hat{j}(-1-0) + \hat{k}(1-0)$

$= \hat{i} + \hat{j} + \hat{k}$

Point on plane $\vec{q} = \hat{i} + 0\hat{j} + 0\hat{k}$

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$(x\hat{i} + y\hat{j} + z\hat{k})(-\hat{i} + \hat{j} + \hat{k}) = -1$



P($1-\lambda$, $0+\lambda$, $1+\lambda$)
P on plane
$\lambda - 1 + \lambda + 1 + \lambda = -1$
$3\lambda = -1$

$\lambda = -\frac{1}{3}$

$\alpha = 1 - \lambda = \frac{4}{3}$

$\beta = -\frac{1}{3}$

$\gamma = 1 - \frac{1}{3} = \frac{2}{3}$

$= (\alpha + \beta + \gamma)$

$= 3\left(\frac{4}{3} - \frac{1}{3} + \frac{2}{3}\right)$

$= 5$