JEE Main September 2020

Question Paper With Text Solution

2 September| Shift-1

MATHEMATICS

JEE MAIN SEP 2020 | 2 SEP SHIFT-1

1. If p(x) be a polynomial of degree three that has a local maximum value 8 at x = 1 and a local minimum value 4 at x = 2 ; then p(0) is equal to :

(1) 6 (2) 12 (3) – 24 (4) – 12

Ans. (4)

Sol. $P'(x) = \lambda(x-1)(x-2)$

$P'(x) = \lambda(x^2 - 3x + 2)$

$P(x) = \lambda\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right) + C$

$P(1) = 8 \Rightarrow \lambda\left(\frac{1}{3} - \frac{3}{2} + 2\right) + C = 8$

$8 = \frac{5\lambda}{6} + C$ .... (1)

$P(2) = 4 \Rightarrow \lambda\left(\frac{8}{3} - 6 + 4\right) + C = 4$

$\frac{2\lambda}{3} + C = 4$ .... (2)

$\lambda = 24$, $C = -12$,

$P(0) = C \Rightarrow P(0) = -12$.

2. A line parallel to the straight line 2x – y = 0 is tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ at the point (x_1, y_1). Then $x_1^2 + 5y_1^2$ is equal to :

(1) 6 (2) 5 (3) 10 (4) 8

Ans. (1)

Sol. Hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$

$P(x_1, y_1) \equiv P(2\sec\theta, \sqrt{2}\tan\theta)$

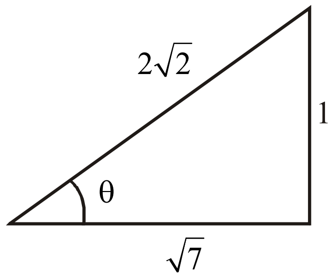
Equation of tangent at P

$$\frac{x(2\sec\theta)}{2}-\frac{y(\sqrt{2}\tan\theta)}{2}=1$$

$$\frac{x\sec\theta}{2}-\frac{y\tan\theta}{\sqrt{2}}=1$$

slope of tangent = 2

$$\sin\theta=\frac{1}{2\sqrt{2}}$$



$$x_1^2+5y_1^2=4\sec^2\theta+10\tan^2\theta$$

$$=4\left(\frac{8}{7}\right)+10\left(\frac{1}{7}\right)$$

$$=6$$

3. Let $\alpha>0, \beta>0$ be such that $\alpha^3+\beta^2=4$. If the maximum value of the term independent of x in the binomial expansion of $\left(\alpha x^{\frac{1}{9}}+\beta x^{-\frac{1}{6}}\right)^{10}$ is 10 k, then k is equal to :

(1) 352 (2) 176 (3) 84 (4) 336

Ans. (4)

Sol. in $\left(\alpha x^{1/9}+\beta x^{-1/6}\right)^{10}$

$$T_{r+1}={}^{10}C_r\left(\alpha x^{1/9}\right)^{10-r}\left(\beta x^{-1/6}\right)^r$$

$$={}^{10}C_r\,\alpha^{10-r}\beta^r x^{\left(\frac{10-r}{9}-\frac{r}{6}\right)}$$

$$\frac{10-r}{9}-\frac{r}{6}=0\Rightarrow r=4$$

$T_5 = {}^{10}C_4\,\alpha^6\beta^4 = 210\alpha^6\beta^4$

Now $\alpha^3 + \beta^2 = 4$

then by $AM \geq GM$

$$\frac{\alpha^3}{2}+\frac{\alpha^3}{2}+\frac{\beta^2}{2}+\frac{\beta^2}{2} \geq 4\left(\frac{\alpha^6\beta^4}{16}\right)^{1/4}$$

$\alpha^6\beta^4 \leq 16$

then $T_5)_{max} = 210 \times 16 = 10\,k$

$k = 336$

4. If the tangent to the curve $y = x + \sin y$ at a pont (a, b) is parallel to the joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right)$, then :

(1) $|a + b| = 1$ (2) $b = a$ (3) $|b-a| = 1$ (4) $b = \frac{\pi}{2} + a$

Ans. (3)

Sol. $y = x + \sin y$ slope of line joining $\left(0, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, 2\right) = 1$

$$\frac{dy}{dx} = 1 + \cos y\frac{dy}{dx}$$

$$\frac{dy}{dx}(1 - \cos y) = 1$$

$$\frac{dy}{dx} = \frac{1}{1-\cos y}$$

$$\left(\frac{1}{1-\cos y}\right)_{(a,b)} = 1$$

$\cos b = 0$

$b - a = \sin b$

$|b - a| = |\sin b|$

$|b - a| = 1$

5. Let $X = \{x \in N : 1 \leq x \leq 17\}$ and $Y = \{ax + b : x \in X \text{ and } a, b \in R, a > 0\}$. If mean and variance of elements of Y are 17 and 216 respectively then a + b is equal to :

(1) 9 (2) – 7 (3) 7 (4) – 27

Ans. (2)

Sol. $X = \{1, 2, ...17\}$

Mean $= \frac{1+2+.....+17}{17} = 9$

Variance $= \frac{\sum x_1^2}{17} - (\bar{x})^2 = 105 - 81 = 24$

Y = $\{ax + b : x \in X \text{ and } a, b \in R, a > 0\}$

Mean = 9a + b = 17

Variance = $24a^2 = 216$

$\Rightarrow a^2 = 9 \Rightarrow a = 3$

b = – 10

6. Let P(h, k) be a point on the curve $y = x^2 + 7x + 2$, nearest to the line, $y = 3x - 3$. Then the equation of the normal to the curve at P is :

(1) $x + 3y - 62 = 0$ (2) $x - 3y + 22 = 0$ (3) $x + 3y + 26 = 0$ (4) $x - 3y - 11 = 0$

Ans. (3)

Sol. $y = x^2 + 7x + 2$

tangent at P will be parallel to $y = 3x - 3$

$\left(\frac{dy}{dx}\right)_P = 3$

$2x + 7 = 3$

$x = -2$

P(–2, –8)

slope of normal at $P = \frac{-1}{3}$

equation of normal at P

$y + 8 = \frac{-1}{3}(x + 2)$

$x + 3y + 26 = 0$

7. The contrapositive of the statement "If I reach the station in time, then I will catch the train" is :

(1) If I will catch the train, then I reach the station in time.

(2) If I do not reach the station in time, then I will catch the train.

(3) If I will not catch the train, then I do not reach the station in time.

(4) If I do not reach the station in time, then I will not catch the train

Ans. (3)

Sol. $p \equiv$ I reach the station in time

$q \equiv$ I will catch the train.

$p \rightarrow q$

$\sim q \rightarrow (\sim p)$

If I will not catch train, then I do not reach the station in time.

8. The value of $\left(\dfrac{1+\sin\frac{2\pi}{9}+i\cos\frac{2\pi}{9}}{1+\sin\frac{2\pi}{9}-i\cos\frac{2\pi}{9}}\right)^3$ is :

(1) $-\frac{1}{2}\left(1-i\sqrt{3}\right)$ (2) $\frac{1}{2}\left(1-i\sqrt{3}\right)$ (3) $-\frac{1}{2}\left(\sqrt{3}-i\right)$ (4) $\frac{1}{2}\left(\sqrt{3}-i\right)$

Ans. (3)

Sol. Let $\frac{2\pi}{9}=\theta$

$z_1 = 1+\sin\theta+i\cos\theta$

$=1+\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)$

$=2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$=2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left(\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)$

$z_1 = 2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}$

$$\left(\frac{z_1}{\bar{z}_1}\right)^3 = \left(e^{i\left(\frac{\pi}{2}-\theta\right)}\right)^3 = e^{i\left(\frac{3\pi}{2}-3\theta\right)}$$

$= e^{i(270^\circ-120^\circ)}$

$= e^{\,i150^\circ}$

$= \cos 150^\circ + i\sin 150^\circ \;= -\frac{\sqrt{3}}{2}+\frac{i}{2}$

9. Let $y = y(x)$ be the solution; of the differential equation, $\frac{2+\sin x}{y+1}.\frac{dy}{dx} - \cos x, y(0) = 1$. If $y(\pi) = a$ and $\frac{dy}{dx}$ at $x = \pi$ is b, then the ordered pair (a, b) is equal to :

(1) $\left(2, \frac{3}{2}\right)$ (2) (2, 1) (3) (1, 1) (4) (1, –1)

Ans. (3)

Sol. $\frac{dy}{y+1} = \frac{-\cos x\, dx}{2+\sin x}$

$l n(y+1) = -l n(2+\sin x) + l n\, c$

$(y+1)(2+\sin x) = c$

$x = 0$ and $y = 1 \Rightarrow c = 4$

$(y+1)(2+\sin x) = 4$

Put $x = \pi$

$(y+1)(2+0) = 4$

$y = 1$

$\Rightarrow a = 1$

$\frac{dy}{dx} = \frac{-\cos x}{(2+\sin x)}(y+1)$

put $x = \pi$, $y = 1$

$b = \frac{(-1)}{2+0}(1+1)$

$= 1$

10. Let α and β be the roots of the equation, $5x^2 + 6x - 2 = 0$. If $S_n = \alpha^n + \beta^n, n = 1,2,3,.....,$ then :

(1) $6S_6 + 5S_6 = 2S_4$ (2) $5S_6 + 6S_5 = 2S_4$

(3) $5S_6 + 6S_5 + 2S_4 = 0$ (4) $6S_6 + 5S_5 + 2S_4 = 0$

Ans. (2)

Sol. $S_n = \alpha^n + \beta^n$

$5\alpha^2 + 6\alpha - 2 = 0$

$\alpha^2 = \frac{2-6\alpha}{5}$; $\beta^2 = \frac{2-6\beta}{5}$

$S_n = \alpha^{n-2}\left(\alpha^2\right) + \beta^{n-2}\left(\beta^2\right)$

$= \alpha^{n-2}\left(\frac{2-6\alpha}{5}\right) + \beta^{n-2}\left(\frac{2-6\beta}{5}\right)$

$5S_n = 2\alpha^{n-2} - 6\alpha^{n-1} + 2\beta^{n-2} - 6\beta^{n-1}$

$5S_n = -6\left(\alpha^{n-1} + \beta^{n-1}\right) + 2\left(\alpha^{n-2} + \beta^{n-2}\right)$

$5S_n = -6S_{n-1} + 2S_{n-2}$

Put n = 6.

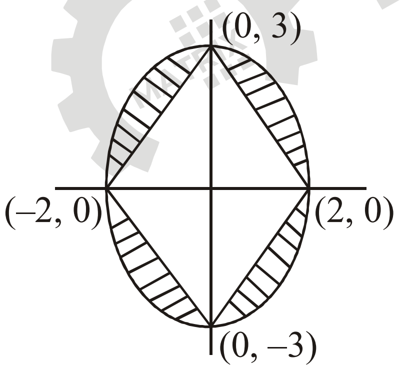
11. Area (in sq. units) of the region outside $\frac{|x|}{2} + \frac{|y|}{3} = 1$ and inside the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is :

(1) $6(4 - \pi)$ (2) $3(\pi - 2)$ (3) $3(4 - \pi)$ (4) $6(\pi - 2)$

Ans. (4)

Sol. The area of region given by

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ & $3|x| + 2|y| = 6$



Required area = area of ellipse – area of rhombus

$= 6\pi - 12$

12. The domain of the function $f(x)=\sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$ is :

$(-\infty, a] \cup [a, \infty)$. Then a is equal to :

(1) $\frac{\sqrt{17}}{2}+1$ (2) $\frac{\sqrt{17}}{2}$ (3) $\frac{1+\sqrt{17}}{2}$ (4) $\frac{\sqrt{17}-1}{2}$

Ans. (3)

Sol. $f(x)=\sin^{-1}\left(\frac{|x|+5}{x^2+1}\right)$

$-1 \le \frac{|x|+5}{x^2+1} \le 1$

$-x^2-1 \le |x|+5$

always true

$|x|+5 \le x^2+1$

put $|x|=t$

$t+5 \le t^2+1$

$t^2-t-4 \ge 0$

$\left(t-\frac{1+\sqrt{17}}{2}\right)\left(t-\frac{1-\sqrt{17}}{2}\right) \ge 0$

$t \ge \frac{1+\sqrt{17}}{2}$ $(\because t \ge 0)$

$|x| \ge \frac{1+\sqrt{17}}{2}$

$x \in \left(-\infty, -\left(\frac{1+\sqrt{17}}{2}\right)\right] \cup \left[\frac{1+\sqrt{17}}{2}, \infty\right)$

$\Rightarrow a=\frac{1+\sqrt{17}}{2}$

13. If $|x| < 1$, $|y| < 1$ and $x \neq y$, then the sum to infinity of the following series

$(x+y)+(x^2+xy+y^2)+(x^3+x^2y+xy^2+y^3)+....$ is :

(1) $\frac{x+y+xy}{(1-x)(1-y)}$ (2) $\frac{x+y-xy}{(1+x)(1+y)}$ (3) $\frac{x+y-xy}{(1-x)(1-y)}$ (4) $\frac{x+y+xy}{(1+x)(1+y)}$

Ans. (3)

Sol. $\frac{x^2-y^2}{x-y}+\frac{x^3-y^3}{x-y}+\frac{x^4-y^4}{x-y}+............\infty$

$$\frac{1}{x-y}\left(x^2+x^3+x^4+........\infty\right)-\frac{1}{x-y}\left(y^2+y^3+y^4+........\right)$$

$$\frac{1}{x-y}\left(\frac{x^2}{1-x}-\frac{y^2}{1-y}\right)=\frac{1}{x-y}\left(\frac{x^2-x^2y-y^2+xy^2}{(1-x)(1-y)}\right)=\frac{x+y-xy}{(1-x)(1-y)}$$

14. Box I contains 30 cards numbered 1 to 30 and Box II contains 20 cards numbered 30 to 50. A box is selected at random and a card is drawn from it. The number on the card is found to be a non-prime number. The probability that the card was drawn from Box I is :

(1) $\frac{2}{5}$ (2) $\frac{2}{3}$ (3) $\frac{8}{17}$ (4) $\frac{4}{17}$

Ans. (3)

Sol. $A \equiv$ Box I is selected

$B \equiv$ Box II is selected

$N \equiv$ non-prime number is selected

$P(N) = P(A)\,P(N/A) + P(B)\,P(N/B)$

$$P(A/N)=\frac{P(A)P(N/A)}{P(A)P(N/A)+P(B)P(N/B)}$$

$$\frac{\frac{1}{2}\times\frac{20}{30}}{\frac{1}{2}\times\frac{20}{30}\times\frac{1}{2}\times\frac{15}{20}}$$

$$=\frac{8}{17}$$

15. If a function f(x) defined by

$$f(x)=\begin{cases} ae^{x}+be^{-x}, & -1\le x<1 \\ cx^{2}, & 1\le x\le 3 \\ ax^{2}+2cx, & 3<x\le 4 \end{cases}$$

be continuous for some a , b, c $\in$ R and $f'(0)+f'(2)=e$, then the value of a is :

(1) $\frac{e}{e^{2}+3e+13}$ (2) $\frac{e}{e^{2}-3e-13}$ (3) $\frac{1}{e^{2}-3e+13}$ (4) $\frac{e}{e^{2}-3e+13}$

Ans. (4)

Sol. $$f(x)=\begin{cases} ae^{x}+be^{-x}, & -1\le x<1 \\ cx^{2}, & 1\le x\le 3 \\ ax^{2}+2cx, & 3<x\le 4 \end{cases}$$

Continuous at x = 1

$ae+\frac{b}{e}=c$ ......(1)

Continuous at x = 3

9c = 9a + 6c

3c = 9a ......(2)

c = 3a

$f'(x)=ae^{x}-be^{x}$ in nbd of x = 0

$f'(0)=a-b$

$f'(x)=2c$ in nbd of x = 2

$f'(2)=4c$

a – b + 4c = e

a – b + 12a = e .....(3)

$ae+\frac{b}{e}=3a$

b = ae (3 – e)

put this in equation (3)

$13a-a(3e-e^{2})=e$

$a = \frac{e}{e^2 - 3e + 13}$

16. The plane passing through the points (1, 2,1), (2, 1, 2) and parallel to the line, 2x = 3y, z = 1 also passes through the point :

(1) (–2, 0, 1) (2) (0, 6, –2) (3) (2, 0, –1) (4) (0, –6, 2)-

Ans. (1)

Sol. P(1, 2, 1) Q(2, 1, 2)

Line $\frac{x}{3} = \frac{y}{2} = \frac{z-1}{0}$

$\vec{m}$ = direction vector of line = (3 , 2, 0)

$\vec{n} = \overrightarrow{PQ} \times \vec{m}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 2\hat{i} - 3\hat{j} - 5\hat{k}$

Equation of plane

17. If $R = \{(x, y) : x, y \in Z, x^2 + 3y^2 \le 8\}$ is a relation on the set of integers Z, then the domain of R^{-1} is :

(1) {–2, –1, 1, 2} (2) {–2, –1, 0, 1, 2} (3) {0, 1} (4) {–1, 0, 1}

Ans. (4)

Sol. Domain of R^{-1} = Range of R

Range of R will contain all possible value of y.

Range of R {0, 1, –1}

18. Let S be the set of all $\lambda \in R$ for which the system of linear equations

$2x - y + 2z = 2$

$x - 2y + \lambda z = -4$

$x + \lambda y + z = 4$

has no solution. Then the set S.

(1) contains more than two elements (2) is a singleton

(3) is an empty set (4) contains exactly two elements

Ans. (4)

Sol. $$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & \lambda \\ 1 & \lambda & 1 \end{vmatrix} = 0$$

$2(-2-\lambda^2) + (1-\lambda) + 2(\lambda+2) = 0$

$-4 - 2\lambda^2 + 1 - \lambda + 2\lambda + 4 = 0$

$2\lambda^2 - \lambda - 1 = 0$

$\lambda = 1$ or $\lambda = -\frac{1}{2}$

If $\lambda = 1$

$2x - y + 2z = 2$

$x - 2y + z = -4$

$x + y + z = 4$

$D_x \neq 0 \Rightarrow$ No solution

If $\lambda = -\frac{1}{2}$

$2x - y + 2z = 2$

$2x - 4y - z = -8$

$2x - y + 2z = 8$

$\Rightarrow$ No solution.

$\Rightarrow S \equiv \left\{+1, \frac{-1}{2}\right\}$

19. Let A be a 2×2 real matrix with entries from $\{0, 1\}$ and $|A| \neq 0$. Consider the following two statements :

(P) If $A \neq I_2$, then $|A| = -1$

(Q) If $|A| = 1$, then $tr(A) = 2$,

where I_2 denotes 2×2 identity matrix and tr(A) denotes the sum of the diagonal entries of A. Then :

(1) Both (P) and (Q) are false
(2) (P) is false and (Q) is true
(3) (P) is true and and (Q) is false
(4) Both (P) and (Q) are true

Ans. (2)

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = ad - bc$

Statement P

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ P is false

Statement Q

$|A| = 1 \quad \Rightarrow ad = 1 \quad \Rightarrow a = 1 \;\&\; d = 1 \;\&\; bc = 0$

$tr(A) = a + d = 2$

Q is true

20. The sum of the first three terms of a G.P is S and their product is 27. Then all such S lie in :

(1) $(-\infty, 9]$
(2) $(-\infty, -9] \cup [3, \infty)$
(3) $(-\infty, -3] \cup [9, \infty)$
(4) $[-3, \infty)$

Ans. (3)

Sol. $\frac{a}{r}, \; a, \; ar$

$\frac{a}{r}.a.ar = 27$

$a = 3$

$S = \frac{a}{r} + a + ar$

$= 3\left(\frac{1}{r} + 1 + r\right)$

$= 3 + 3\left(r + \frac{1}{r}\right)$

$3 + 3\ (-\infty, -2] \cup [2, \infty)$

$3 + (-\infty, -6] \cup [6, \infty)$

$(-\infty, -3] \cup [9, \infty)$

21. If the letters of the word 'MOTHER' be permuted and all the words so formed (with or without meaning) be listed as in a dictionary, then the position of the word 'MOTHER' is ______.

Ans. 309

Sol. Rank of 'MOTHER'

E — 120

H — 120

ME — 24

MH — 24

MOE — 6

MOH — 6

MOR — 6

MOTE — 2

MOTHER 1

Sum = 309

22. If $\lim_{x \to 1} \frac{x + x^2 + x^3 + ... + x^n - n}{x - 1} = 820, (n \in N)$, then the value of n is equal to _____.

Ans. 40

Sol. $\lim_{x \to 1} \frac{x-1}{x-1} + \frac{x^2-1}{x-1} + ... + \frac{x^n - 1}{x-1} = 820$

$1 + 2 + ... + n = 820$

$$\frac{n(n+1)}{2} = 820$$

$$n^2 + n - 1640 = 0$$

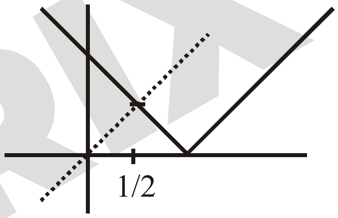
$$(n+41)(n-40) = 0$$

$$\Rightarrow n = 40$$

23. The integral $\int_0^2 \| x-1| - x \mid dx$ is equal to ______.

Ans. 1.5

Sol. $\int_0^2 \| x-1| - x \mid dx$



$$\int_0^{1/2} (|x-1| - x)dx + \int_{1/2}^{2} (x - |x-1|)dx$$

$$\int_0^{1/2} (1-x-x)dx + \int_{1/2}^{1} (x-(1-x))dx + \int_1^2 (x-(x-1))dx$$

$$\left(x - x^2\right)_0^{1/2} + \left(x^2 - x\right)_{1/2}^{1} + (x)_1^2$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(0 - \left(\frac{1}{4} - \frac{1}{2}\right) + (2-1)\right)$$

$$= \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$$

24. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be the three unit vectors such that $|\vec{a} - \vec{b}|^2 + |\vec{a} - \vec{c}|^2 = 8$. Then $|\vec{a} + \vec{b}|^2 + |\vec{a} + 2\vec{c}|^2$ is equal to _____.

Ans. 2

Sol. $|\hat{a} - \hat{b}|^2 + |\hat{a} - \hat{c}|^2 = 8$

$$1 + 1 - 2\hat{a}.\hat{b} + 1 + 1 - 2\hat{a}.\hat{c} = 8$$

$$\hat{a}.\hat{b} + \hat{a}.\hat{c} = -2$$

$$E = \left|\hat{a} + 2\hat{b}\right|^2 + \left|\hat{a} + 2\hat{c}\right|^2$$

$$= 1 + 4 + 4\hat{a}.\hat{b} + 1 + 4 + 4\hat{a}.\hat{c}$$

$$= 10 + 4(\hat{a}.\hat{b} + \hat{a}.\hat{c})$$

$= 10 + 4(-2) = 10 - 8 = 2.$

25. The number of integral values of k for which the line, $3x + 4y = k$ intersects the circle,

$x^2 + y^2 - 2x - 4y + 4 = 0$ at two distinct points is ____.

Ans. 9

Sol. Centre $\equiv (1, 2)$ $r = 1$

If line intersects the circle at two distinct points then its distance from centre will be less than radius.

$$\frac{|3+8-k|}{5} < 1$$

$|k - 11| < 5$

$-5 < k - 11 < 5$

$6 < k < 16$