JEE Adv. May 2024

Question Paper With Text Solution

26 May | Paper-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911

Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE ADV. MAY 2024 | 26$^{TH.}$ MAY PAPER-2

SECTION – 1 (MAXIMUM MARKS: 12)

- This section contains **FOUR (04)** question stems.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –1 In all other cases.

1. Considering only the principal values of the inverse trigonometric functions, the value of $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is :

(A) $\frac{7}{24}$ (B) $\frac{-7}{24}$ (C) $\frac{-5}{24}$ (D) $\frac{5}{24}$

Ans. B

Sol. $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

$\sin^{-1}\left(\frac{3}{5}\right)=\theta$ $2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)=\phi$

$\sin\theta=\frac{3}{5}$ $\cos\frac{\phi}{2}=\frac{2}{\sqrt{5}}$

$\tan\theta=\frac{3}{4}$ $\tan\frac{\phi}{2}=\frac{1}{2}$

$$\tan\phi=\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}$$

$$\tan\phi = \frac{2\left(\frac{1}{2}\right)}{1-\frac{1}{4}}$$

$$\tan\phi = \frac{4}{3}$$

$\tan(\phi - \theta)$

$\dfrac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

$\dfrac{\frac{3}{4}-\frac{4}{3}}{1+1} = \dfrac{9-16}{24}$

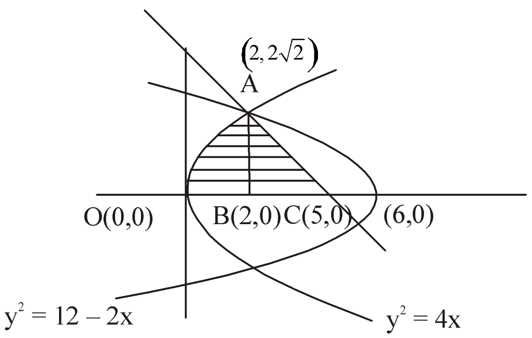
$= -\dfrac{7}{24}$

2. Let $S = \left\{(x, y) \in R \times R : x \geq 0, y \geq 0, y^2 \leq 4x, y^2 \leq 12 - 2x \text{ and } 3y + \sqrt{8}x \leq 5\sqrt{8}\right\}$ If the area of the region S is $\alpha\sqrt{2}$, then α is eqaul to :

(A) $\dfrac{17}{2}$ (B) $\dfrac{17}{3}$ (C) $\dfrac{17}{4}$ (D) $\dfrac{17}{5}$

Ans. B

Sol. $3y + \sqrt{8}x = 5\sqrt{8}$



$4x = 12 - 2x \Rightarrow 6x = 12 \Rightarrow x = 2$

$x = 2 \Rightarrow y^2 = 8 \Rightarrow y = \pm 2\sqrt{2}$

line $3y + \sqrt{8}x = 5\sqrt{8}$ passes through $A\left(2, 2\sqrt{2}\right)$ & C(5,0)

Required area = Area of OAB + Area of ΔABC

$$= \int_0^2 2\sqrt{x}\,dx + \frac{1}{2}(5-2)\left(2\sqrt{2}\right)$$

$$= \frac{4}{3}\left(x^{\frac{3}{2}}\right)_0^2 + 3\sqrt{2}$$

$$= \frac{8}{3}\sqrt{2} + 3\sqrt{2}$$

$$= \frac{17}{3}\sqrt{2}$$

3. Let $k \in R$. If $\lim_{x \to 0+} \left(\sin(\sin kx) + \cos x + x\right)^{\frac{2}{x}} = e^6$, then the value of k is :

(A) 1 (B) 2 (C) 3 (D) 4

Ans. B

Sol. $$\lim_{x \to 0^+} \left(\sin(\sin kx) + \cos x + x\right)^{\frac{2}{x}} = e^6$$

$$e^{\lim_{x \to 0^+} 2\left\{\frac{\sin(\sin kx) + \cos x + x - 1}{x}\right\}} = e^6$$

$$\lim_{x \to 0^+} \frac{\sin(\sin kx) + \cos x + x - 1}{x} = 3$$

$$\lim_{x \to 0^+} \frac{\cos(\sin kx) \cdot \cos(kx) \cdot k - \sin x + 1}{1} = 3$$

$k + 1 = 3 \Rightarrow k = 2$

4. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x^2}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$. Then which of the following statements is TRUE?

(A) $f(x) = 0$ has infinitely many solutions in the interval $\left[\frac{1}{10^{10}}, \infty\right)$

(B) $f(x) = 0$ has no solutions in the interval $\left[\frac{1}{\pi}, \infty\right)$

(C) The set of solutions of $f(x) = 0$ in the interval $\left(0, \frac{1}{10^{10}}\right)$ is finite

(D) $f(x) = 0$ has more than 25 solutions in the interval $\left(\frac{1}{\pi^2}, \frac{1}{\pi}\right)$

Ans. D

Sol. for $x > 0$

$$f(x) = 0 \Rightarrow x^2 \sin\left(\frac{\pi}{x^2}\right) = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{x^2}\right) = 0$$

$$\Rightarrow \frac{\pi}{x^2} = n\pi$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{n}}$$

$$x \in \left(\frac{1}{\pi^2}, \frac{1}{\pi}\right) \Rightarrow \frac{1}{\pi^2} < \frac{1}{\sqrt{n}} < \frac{1}{\pi}$$

$$\pi < \sqrt{n} < \pi^2$$

$$\pi^2 < n < \pi^4$$

$f(x) = 0$ has more than 25 solutions

SECTION – 2 (MAXIMUM MARKS: 12)

- This section contains **THREE (03)** question stems.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 **ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;
Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;
Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is acorrect option;
Zero Marks : 0 If unanswered;
Negative Marks : –2 In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get -2 marks.

5. Let S be the set of all $(\alpha, \beta) \in R \times R$ such that $\lim_{x\to\infty} \dfrac{\sin(x^2)(\log_e x)^{\alpha} \sin\left(\frac{1}{x^2}\right)}{x^{\alpha\beta}(\log_e(1+x))^{\beta}} = 0$. Then which of the following is (are) correct?

(A) $(-1, 3) \in S$ (B) $(-1, 1) \in S$ (C) $(1, -1) \in S$ (D) $(1, -2) \in S$

Ans. BC

Sol. $$\lim_{x\to\infty} \frac{\sin\left(x^2\right)\left(\ell n(x)\right)^{\alpha} \sin\left(\frac{1}{x^2}\right)}{x^{\alpha\beta}\left(\ell n(1+x)\right)^{\beta}} = 0$$

$$\lim_{x\to\infty} \frac{\sin\left(x^2\right)\ell n(x)^{\alpha}\left(\frac{1}{x^2}\right)}{x^{\alpha\beta}\left(\ell n(1+x)\right)^{\beta}} = 0$$

$$\lim_{x\to\infty} \frac{\sin\left(x^2\right)\left(\ell nx\right)^{\alpha}}{x^{\alpha\beta+2}\left(\ell n(1+x)\right)^{\beta}} = 0$$

$\Rightarrow \alpha\beta + 2 > 0$ or $\alpha\beta + 2 = 0$ & $\beta > \alpha$

$\alpha\beta > -2$ $\alpha\beta = -2$ & $\beta > \alpha$

(A) $\alpha\beta = -3$ Incorrect

(B) $\alpha\beta = -1$ Correct

(C) $\alpha\beta = -1$ Correct

(D) $\alpha\beta = -2$ but $\alpha > \beta$ Incorrect

6. A straight line drawn from the point P(1, 3, 2), parallel to the line $\frac{x-2}{1} = \frac{y-4}{2} = \frac{z-6}{1}$, intersects the plane $L_1 : x - y + 3z = 6$ at the point Q. Another straight line which passes through Q and is perpendicular to the plane L_1 intersects the plane $L_2 : 2x - y + z = -4$ at the point R. Then which of the following statements is (are) TRUE?

(A) The length of the line segment PQ is $\sqrt{6}$

(B) The coordinates of R are (1, 6, 3)

(C) The centroid of the triangle PQR is $\left(\frac{4}{3}, \frac{14}{3}, \frac{5}{3}\right)$

(D) The perimeter of the triangle PQR is $\sqrt{2} + \sqrt{6} + \sqrt{11}$

Ans. AC

Sol. P(1,3,2)

$\vec{r} = (1,3,2) + \lambda(1,2,1)$

Line intersects plane L_1 at Q

Let Q $(1 + \lambda, 3 + 2\lambda, 2 + \lambda)$

$(1 + \lambda) - (3 + 2\lambda) + 3(2 + \lambda) = 6$

$\Rightarrow \lambda = 1$

$\Rightarrow$ Q(2,5,3)

equations of line through Q

$\vec{r} = (2,5,3) + \mu(1,-1,3)$

Let $R(2 + \mu, 5 - \mu, 3 + 3\mu)$

R will lie on plane L_2

$2(2 + \mu) - (5 - \mu) + 3 + 3\mu = -4$

$\Rightarrow \mu = -1$

$\Rightarrow$ R(1,6,0)

(A) $PQ = \sqrt{6}$

(B) R(1,6,0)

(C) Centroid opf $\Delta PQR \equiv \left(\frac{4}{3}, \frac{14}{3}, \frac{5}{3}\right)$

(D) Perimeter of $\Delta PQR = \sqrt{6} + \sqrt{11} + \sqrt{13}$

7. Let A_1, B_1, C_1 be three points in the xy-plane. Suppose that the lines A_1C_1 and B_1C_1 are tangents to the curve $y^2 = 8x$ at A_1 and B_1, respectively. If O = (0, 0) and $C_1 = (-4, 0)$, then which of the following statements is (are) TRUE? :

(A) The length of the line segment OA_1 is $4\sqrt{3}$

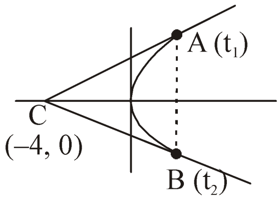
(B) The length of the line segment A_1B_1 is 16

(C) The orthocenter of the triangle $A_1B_1C_1$ is (0, 0)

(D) The orthocenter of the triangle $A_1B_1C_1$ is (1, 0)

Ans. AC

Sol.



tangent

$ty = x + at^2$

$ty = x + 2t^2$

(–4,0) lies on it.

$O = -4 + 2t^2$

$\Rightarrow t^2 = 2 \Rightarrow t = \pm\sqrt{2}$

$A_1\left(at_1^2, 2at_1\right) \equiv A_1\left(4, 4\sqrt{2}\right)$

$B_1\left(at_2^2, 2at_2\right) \equiv B_1\left(4, -4\sqrt{2}\right)$

(A) $OA_1 = 4\sqrt{3}$

(B) $A_1B_1 = 8\sqrt{2}$

(C) Orthocenter.

equation of altitude through A

$y - 4\sqrt{2} = \sqrt{2}\left(x - 4\right)$

$\Rightarrow y = 4\sqrt{2}x$

equation of altitude through C $\qquad y = 0$

$\Rightarrow$ orthocenter $\equiv$ (0,0)

SECTION – 3 (MAXIMUM MARKS: 24)

- This section contains **SIX (06)** question stems.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If **ONLY** the correct integer is entered;

Zero Marks : 0 In all other cases.

8. Let $f : R \to R$ be a fuction such that $f(x+y) = f(x) + f(y)$ for all $x, y \in R$, and $g : R \to (0, \infty)$ be a function such that $g(x + y) = g(x)\, g(y)$ for all $x, y \in R$. If $f\left(\frac{-3}{5}\right) = 12$ and $g\left(\frac{-1}{3}\right) = 2$, then the value of $\left(f\left(\frac{1}{4}\right) + g(-2) - 8\right) g(0)$ is _______.

Ans. 51

Sol. $f : R \to R$

$f(x+y) = f(x) + f(y)$ for all $x, y \in R$

$\Rightarrow f(x) = ax$

Now $f\left(\frac{-3}{5}\right) = 12$

$a = -20$

$\therefore f(x) = -20x$

Again, $g(x+y) = g(x).\, g(y)$ $x, y \in R$.

$\Rightarrow f(x) = k^x$

Now $g\left(\frac{-1}{3}\right) = 2$

$2 = k^{\frac{-1}{3}}$

$\Rightarrow \left(2^{-1}\right)^3 = k$

$k = \frac{1}{8}$

$\therefore\ g(x) = \left(\frac{1}{8}\right)^x$

Now, $\left(f\left(\frac{1}{4}\right) + g(-2) - 8\right) g(0)$

$\Rightarrow [-5 + 64 - 8] . 1 \quad \Rightarrow 64 - 13 \quad \Rightarrow 51$

9. A bag contains N balls out of which 3 balls are white, 6 balls are green, and the remaining balls are blue. Assume that the balls are identical otherwise. Three balls are drawn randomly one after the other without replacement. For i =1, 2, 3, let W_i, G_i, and B_i denote the events that the ball drawn in the i^{th} draw is a white ball, green ball, and blue ball, respectively. If the probability $P\left(W_1 \cap G_2 \cap B_3\right) = \frac{2}{5N}$ and the conditional probability $P\left(B_3 \mid W_1 \cap G_2\right) = \frac{2}{9}$, then N equals _______.

Ans. 11

Sol. $3 \to W$

$6 \to G$

Let total balls be N

So remaining balls will be $N - 9$

$N-9 \to$ Blue

Now, $P\left(W_1 \cap G_2 \cap B_3\right) = \frac{2}{5N}$

$\frac{3}{N} \times \frac{6}{N-1} \times \frac{N-9}{N-2} = \frac{2}{5N}$

$\frac{18}{N-1} \times \frac{N-9}{N-2} = \frac{2}{5}$

$45\ (N-9) = (N-1)\ (N-2)$

$45N - (45 \times 9) = N^2 - 3N + 2$

$N^2 - 48N + 407 = 0$

$(N - 11)\ (N - 37) = 0$

$\Rightarrow N - 11$ or $N = 37$ ________(A)

Again $P\left(\frac{B_3}{W_1 \cap G_2}\right) = \frac{2}{9}$

$\frac{N-9}{N-2} = \frac{2}{9}$

$9N - 81 = 2N - 4$

$7N = 77$

$\Rightarrow N = 11$ ________(B)

from A and B $\boxed{N = 11}$

10. Let the function $f : R \to R$ be defined by $f(x) = \frac{\sin x}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{\left(x^2 - x + 3\right)} + \frac{2}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{\left(x^2 - x + 3\right)}$

Then the number of solutions of $f(x) = 0$ in R is_______.

Ans. 1

Sol. $f(x) = \frac{\sin x}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{\left(x^2 - x + 3\right)} + \frac{2}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{\left(x^2 - x + 3\right)}$

$\Rightarrow \frac{\sin x}{e^{\pi x}} \frac{\left(x^{2023} + 2024x + 2025\right)}{\left(x^2 - x + 3\right)} [\sin x + 2] = 0$

Now $\sin x + 2 = 0$

$\sin x = -2$ (No sol.)

So we need to solve

$x^{2023} + 2024x + 2025 = 0$

Let $g(x) = x^{2023} + 2024x + 2025$

Differentiate

$2023x^{2022}+2024$ (Always positive)

So It cuts x -axis only once.

So exactly 1 solution.

11. Let $\vec{p}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{q}=\hat{i}-\hat{j}+\hat{k}$. If for some real numbers α, β, and γ, we have $15\hat{i}+10\hat{j}+6\hat{k}=\alpha(2\vec{p}+\vec{q})+\beta(\vec{p}-2\vec{q})+\gamma(\vec{p}\times\vec{q})$ then the value of γ is _______.

Ans. 2

Sol. $\vec{p}=2\hat{i}+\hat{j}+3\hat{k}$

$\vec{q}=\hat{i}-\hat{j}+\hat{k}$ d

$15\hat{i}+10\hat{j}+6\hat{k}=\alpha(2\vec{p}+\vec{q})+\beta(\vec{p}-2\vec{q})+\gamma(\vec{p}\times\vec{q})$

Take dot with $\vec{p}\times\vec{q}$

$\left(15\hat{i}+10\hat{j}+6\hat{k}\right).(\vec{p}\times\vec{q})=\alpha\left(2\vec{p}+\vec{q}\right).(\vec{p}\times\vec{q})+\beta\left(\vec{p}-2\vec{q}\right).(\vec{p}\times\vec{q})+\gamma\left(\vec{p}\times\vec{q}\right).(\vec{p}\times\vec{q})$

$=0+0+\gamma|\vec{p}\times\vec{q}|^2$

Now, $\vec{p}\times\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 3\\ 1 & -1 & 1\end{vmatrix}$

$=\hat{i}(4)-\hat{j}(-1)+\hat{k}(-3)$

$=4\hat{i}+\hat{j}-3\hat{k}$

$|\vec{p}\times\vec{q}|=\sqrt{16+1+9}$

$\Rightarrow\sqrt{26}$

$\left(15\hat{i}+10\hat{j}+6\hat{k}\right)\left(4\hat{i}+\hat{j}-3\hat{k}\right)=\gamma 26$

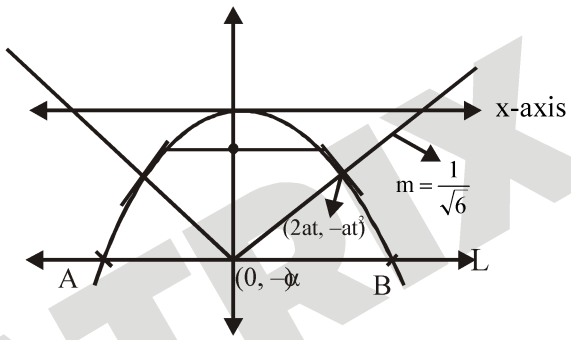
$\dfrac{60+10-18}{26}=\gamma$

$\gamma=2$

12. A normal with slope $\frac{1}{\sqrt{6}}$ is drawn from the point $(0, -\alpha)$ to the parabola $x^2 = -4ay$, where $a > 0$. Let L be the line passing through $(0, -\alpha)$ and parallel to the directrix of the parabola. Suppose that L intersects the parabola at two points A and B. Let r denote the length of the latus rectum and s denote the square of the length of the line segment AB. If r : s = 1 : 16, then the value of 24 a is _______.

Ans. 12

Sol. $x^2 = -4ay$ $(a > 0)$



$r = 4a$

$S = |AB|^2$

$\frac{r}{s} = \frac{1}{16}$

$\Rightarrow 16r = s$

Now $x^2 = -4ay$

$2x = -4a\frac{dy}{dx}$

$\left.\frac{dy}{dx}\right| = -\frac{x}{2a}$

$(2at, -at^2)$

$= -\frac{2at}{2a}$

$= -t$

$m_{tangent} = -t$

$m_{Normal} = \frac{1}{t}$

Given : $\frac{1}{t} = \frac{1}{\sqrt{6}} \Rightarrow t = \sqrt{6}$

Equation of Normal

$(y + at^2) = \frac{1}{\sqrt{6}}(x - 2at)$

$(y + 6a) = \frac{1}{\sqrt{6}}(x - 2\sqrt{6}a)$

$y\sqrt{6} + 6\sqrt{6}a = x - 2\sqrt{6}a$

$x - y\sqrt{6} = 8\sqrt{6}a$

Cuts y-axis at $(0, -\alpha)$

$$0+\alpha\sqrt{6}=8\sqrt{6}a$$

$$\alpha = 8a$$

Now equation of L $\Rightarrow y = -\alpha$

$y = -8a$

For coordinates of A & B

$x^2 = (4a)(8a)$

$x^2 = 32a^2$

$$A\left(-4\sqrt{2}a,-8a\right) \& B\left(4\sqrt{2}a,-8a\right)$$

$$s = |AB| = 8\sqrt{2}a$$

$$s = |AB|^2 = 128$$

Now, $16.4a = 128a^2$

$$a = \frac{1}{2}$$

$$\therefore 24a \Rightarrow 24\times\frac{1}{2}$$

$= 12$

13. Let the function $f:[1,\infty)\to R$ be defined by

$$f(t)=\begin{cases}(-1)^{n+1}2, & \text{if } t=2n-1, n\in N\\ \dfrac{(2n+1-t)}{2}f(2n-1)+\dfrac{(t-(2n-1))}{2}f(2n+1), & \text{if } 2n-1<t<2n+1, n\in N\end{cases}$$

Define $g(x)=\int_1^x f(t)dt, x\in(1,\infty)$. Let α denote the number of solutions of the equation $g(x) = 0$ in the interval $(1, 8]$ and $\beta = \lim\limits_{x\to1+}\dfrac{g(x)}{x-1}$. Then the value of $\alpha+\beta$ is equal to _______.

Ans. 5

Sol. If $t = 2n-1, n \in N$

$f(t) = (-1)^{n+1}.2$

$\Rightarrow f(1) = 2, f(3) = -2, f(15) = 2, f(7) = -2, f(9) = 2,$ ....

If $t \in (2n-1, 2n+1)$

$$f(t)=\frac{(2n+1-t)}{2}f(2n-1)+\frac{(t-(2n-1))}{2}f(2n+1)$$

Now $n = 1$

$1 < t < 3$

$$f(t) = \frac{(3-t)}{2}(2) + \frac{(t-1)}{2}(-2)$$

$\Rightarrow 3 - t - t + 1$

$\Rightarrow 4 - 2t$

now $n = 2$

$3 < t < 5$

$$f(t) = \frac{(5-t)}{2}(-2) + \left(\frac{t-3}{2}\right)(2)$$

$= -5 + t + t - 3$

$\Rightarrow -8 + 2t$

$n = 3$

$5 < t < 7$

$$f(t) = \left(\frac{7-t}{2}\right)(2) + \left(\frac{t-5}{2}\right)(-2)$$

$= 7 - t - t + 5$

$\Rightarrow 12t - 2t$

$n = 4$

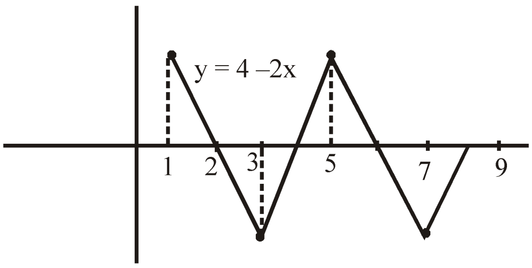
$7 < t < 9$

$$f(t) = \left(\frac{9-t}{2}\right)(-2) + \left(\frac{t-7}{2}\right)(2)$$

$= t - 9 + t - 7$

$\Rightarrow 2t - 16$

Plot f(x)



No. of solution of $g(x) = 3 = \alpha$

$$\beta = \lim_{x\to 1^+} \frac{g(x)}{x-1}$$

$= \lim_{x \to 1^+} g'(x)$

$\Rightarrow f(x) \Rightarrow f(1^+) \Rightarrow 2$

$\alpha + \beta$

$\Rightarrow 3 + 2$

$\Rightarrow 5$

SECTION – 4 (MAXIMUM MARKS: 12)

- This section contains **TWO (02)** paragraphs.
- Based on each paragraph, there are **TWO (02)** questions.
- The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct numerical value is entered in the designated place;

Zero Marks : 0 In all other cases.

PARAGRAPH "I"

Paragraph for Q. No 14 to 15

Let $S = \{1, 2, 3, 4, 5, 6\}$ and X be the set of all relations R from S to S that satisfy both the following properties :

i. R has exactly 6 elements.

ii. For each $(a, b) \in R$, we have $|a - b| \geq 2$.

Let $Y = \{R \in X : \text{The range of R has exactly one element}\}$ and

$Z = \{R \in X : \text{R is a function from S to S}\}$.

Let n(A) denote the number of elements in a set A.

(There are two questions based on PARAGRAPH "I")

14. If $n(X) = {}^{m}C_6$, then the value of m is _______.

Ans. 20

15. If the value of n(Y) + n(Z) is k^2, then |k| is _______.

Ans. 36

Sol. All possible ordered pair (a, b) s.t. $|a - b| \geq 2$

(1, 3)	(2, 4)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 4)	(2, 5)	(3, 5)	(4, 2)	(5, 2)	(6, 2)
(1, 5)	(2, 6)	(3, 6)	(4, 6)	(5, 3)	(6, 3)
(1, 6)					(6, 4)

14. $n(X) = {}^{20}C_6 \Rightarrow m = 20$

15. $n(Y) = 0$ as maximum 4 ordered pairs are possible having same image

$n(Z) = 4 \times 3 \times 3 \times 3 \times 3 \times 4 = 1296$

$n(Y) + n(Z) = 1296 = (36)^2$

$\Rightarrow |k| = 36$

PARAGRAPH "II"

Paragraph for Q. No 16 to 17

Let $f : \left[0, \frac{\pi}{2}\right] \to [0,1]$ be the function defined by $f(x) = \sin^2 x$ and let $g : \left[0, \frac{\pi}{2}\right] \to [0, \infty)$ be the function defined by $g(x) = \sqrt{\frac{\pi x}{2} - x^2}$.

(There are two questions based on PARAGRAPH "II")

16. The value of $2\int_0^{\frac{\pi}{2}} f(x)g(x)dx - \int_0^{\frac{\pi}{2}} g(x)dx$ is _______.

Ans. 0

Sol. Let $I_1 = 2\int\limits_0^{\frac{\pi}{2}} \sin^2 x\sqrt{\frac{\pi x}{2} - x^2}\,dx$ _______(1)

King

$I_1 = 2\int\limits_0^{\frac{\pi}{2}} \cos^2 x\sqrt{\frac{\pi x}{2} - x^2}\,dx$ _______(2)

(1) + (2)

$$2I_1 = 2\int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi x}{2} - x^2}\,dx$$

$$I_1 = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi x}{2} - x^2}\,dx$$

Let $I_2 = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi x}{2} - x^2}\,dx$

Ans. $= I_1 - I_2$

$= 0$

17. The value of $\frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} f(x)g(x)dx$ is _______.

Ans. 0.25

Sol. $I = \frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} \sin^2 x\sqrt{\frac{\pi x}{2} - x^2}\,dx$ ______(1)

Apply king property

$I = \frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} \cos^2 x\sqrt{\frac{\pi x}{2} - x^2}\,dx$ ______(2)

(1) + (2)

$$2I = \frac{16}{\pi^3}\int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi x}{2} - x^2}\,dx$$

$$I = \frac{8}{\pi^3}\int_0^{\frac{\pi}{2}} \sqrt{\frac{\pi^2}{16} - \left(x - \frac{\pi}{4}\right)^2}\,dx$$

Put $x - \frac{\pi}{4} = t$

$dx = dt$

$$I = \frac{8}{\pi^3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{\frac{\pi^2}{16} - t^2}\,dt$$

Even function

$$I = \frac{16}{\pi^3}\int_{0}^{\frac{\pi}{4}} \sqrt{\frac{\pi^2}{16} - t^2}\,dt$$

$$= \frac{16}{\pi^3}\left[t\sqrt{\frac{\pi^2}{16} - t^2} + \frac{\pi^2}{32}\sin^{-1}\left(\frac{t}{\frac{\pi}{4}}\right)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{16}{\pi^3} \times \frac{\pi^2}{32} \times \frac{\pi}{2} = \frac{1}{4} = 0.25$$