JEE Adv. June 2023

Question Paper With Text Solution

04 June | Paper-1

MATHEMATICS

JEE ADV. JUNE 2023 | 4TH. JUNE PAPER-1

SECTION – A

- This section contains **THREE (03)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks :	+4 **ONLY** if (all) the correct option(s) is(are) chosen;
Partial Marks :	+3 If all the four options are correct but **ONLY** three options are chosen;
Partial Marks :	+2 If three or more options are correct but **ONLY** two options are chosen, both of which are correct;
Partial Marks :	+1 If two or more options are correct but **ONLY** one option is chosen and it is a correct option;
Zero Marks :	0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks :	–2 In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

choosing ONLY (A), (B) and (D) will get +4 marks;

choosing ONLY (A) and (B) will get +2 marks;

choosing ONLY (A) and (D) will get +2 marks;

choosing ONLY (B) and (D) will get +2 marks;

choosing ONLY (A) will get +1 mark;

choosing ONLY (B) will get +1 mark;

choosing ONLY (D) will get +1 mark;

choosing no option (i.e. the question is unanswered) will get 0 marks; and

choosing any other combination of options will get –2 marks.

1. Let $S=(0,1)\cup(1,2)\cup(3,4)$ and $T=\{0,1,2,3\}$. Then which of the following statements is(are) true?

(A) There are infinitely many functions from S to T

(B) There are infinitely many strictly increasing functions from S to T

(C) The number of continuous functions from S to T is at most 120

(D) Every continuous function from S to T is differentiable

Ans. ACD

Sol. $S=(0,1)\cup(1,2)\cup(3,4)$

$T=\{0,1,2,3\}$

Number of functions

Each element of S have 4 choice

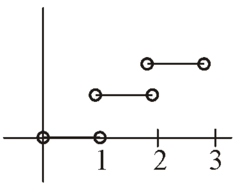
Let n be the number of element in set S.

Number of function $=4^n$

Here $n\to\infty$

$\Rightarrow$ Option (A) is correct.

Option (B) is incorrect (obvious)

(C) 

For continuous function

Each interval will have 4 choices

$\Rightarrow$ Number of continuous functions

$=4\times4\times4=64$

$\Rightarrow$ Option (C) is correct.

(D) Every continuous function is piecewise constant functions

$\Rightarrow$ Differentiable.

Option (D) is correct.

2. Let T_1 and T_2 be two distinct common tangents to the ellipse $E:\frac{x^2}{6}+\frac{y^2}{3}=1$ and the parabola $P:y^2=12x$.

Suppose that the tangent T_1 touches P and E at the points A_1 and A_2, respectively and the tangent T_2 touches P and E at the points A_4 and A_3, respectively. Then which of the following statements is(are) true?

(A) The area of the quadrilateral $A_1A_2A_3A_4$ is 35 square units

(B) The area of the quadrilateral $A_1A_2A_3A_4$ is 36 square units

(C) The tangents T_1 and T_2 meet the x-axis at the point $(-3,0)$

(D) The tangents T_1 and T_2 meet the x-axis at the point $(-6,0)$

Ans. AC

Sol. $E : \frac{x^2}{6} + \frac{y^2}{3} = 1$, Tangent : $y = m_1 x \pm \sqrt{6m_1^2 + 3}$

$P : y^2 = 12x$, Tangent : $y = m_2 x + \frac{3}{m_2}$

For common tangent $m = m_1 = m_2$

$\pm\sqrt{6m^2 + 3} = \frac{3}{m}$

$\Rightarrow m = \pm 1$

$\Rightarrow$ equation of common tangents are $y = x + 3$ and $y = -x - 3$

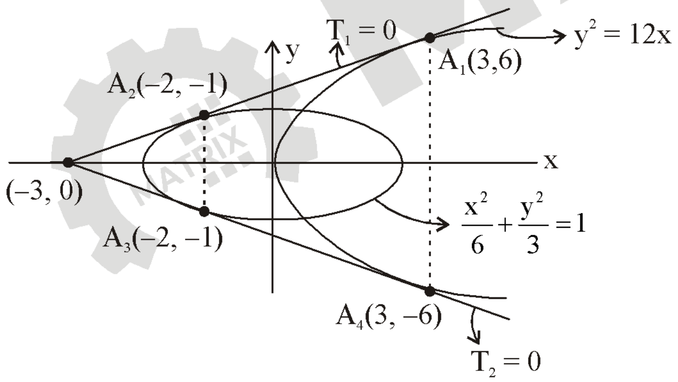
point of contact for parabola is $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$\Rightarrow A_1 \equiv (3,6)$, $A_4 (3 - 6)$

Let $A_2 (x_1, y_1) \Rightarrow$ tangent to E is $\frac{xx_1}{6} + \frac{yy_1}{3} = 1$

$A_2 (x_1, y_1) = A_2 (-2, 1)$

A_3 is mirror image of A_2 in x–axis $\Rightarrow A_3 (-2, -1)$



Intersection point of $T_1 = 0$ and $T_2 = 0$ is $(-3, 0)$

Area of quadrilateral $A_1A_2A_3A_4 = \frac{1}{2}(12 + 2) \times 5 = 35$ square units

3. Let $f : [0,1] \to [0,1]$ be the function defined by $f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$. Consider the square region $S = [0,1] \times [0,1]$. Let $G = \{(x, y) \in S : y > f(x)\}$ be called the green region and $R = \{(x, y) \in S : y < f(x)\}$ be called the red region. Let $L_h = \{(x, h) \in S : x \in [0,1]\}$ be the horizontal line drawn at a height $h \in [0,1]$. Then which of the following statements is(are) true?

(A) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h

(B) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h

(C) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h

(D) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h

Ans. BCD

Sol. $f(x) = \frac{x^3}{3} - x^2 + \frac{5x}{9} + \frac{17}{36}$

$f'(x) = x^2 - 2x + \frac{5}{9}$

$f'(x) = 0$ at $x = \frac{1}{3}$ in $[0,1]$

A_R = Area of Red region
A_G = Area of Green region

$A_R = \int_0^1 f(x)\,dx = \frac{1}{2}$

Total area = 1

$\Rightarrow A_G = \frac{1}{2}$

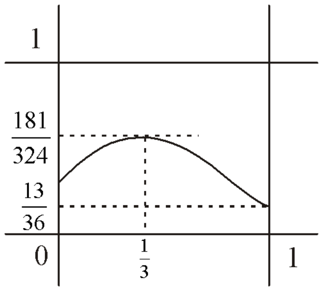
$\int_0^1 f(x)\,dx = \frac{1}{2}$

$A_G = A_R$

$f(0) = \frac{17}{36}$

$f(1) = \frac{13}{36} \approx 0.36$

$f\left(\frac{1}{3}\right) = \frac{181}{324} \approx 0.558$



(A) Given statement would be correct when $h = \frac{3}{4}$ but $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$

$\Rightarrow$ (A) is incorrect

(B) Given statement would be correct when $h = \frac{1}{4}$

$\Rightarrow$ (B) is correct

(C) When $h = \frac{181}{324}, A_R = \frac{1}{2}, A_G < \frac{1}{2}$

$h = \frac{13}{36}, A_R < \frac{1}{2}, A_G = \frac{1}{2}$

$\Rightarrow A_R = A_G$ for some $h \in \left(\frac{13}{36}, \frac{181}{324}\right)$

$\Rightarrow$ (C) is correct

(D) Option (D) is remaining coloured part of option (C), hence option (D) is also correct.

SECTION 2 (Maximum Marks: 12)

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If **ONLY** the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –1 In all other cases.

4. Let $f:(0,1)\rightarrow R$ be the function defined as $f(x)=\sqrt{n}$ if $x\in\left[\frac{1}{n+1},\frac{1}{n}\right)$ where $n\in N$. Let $g:(0,1)\rightarrow R$ be a function such that $\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt<g(x)<2\sqrt{x}$ for all $x\in(0,1)$. Then $\lim_{x\to 0}f(x)g(x)$

(A) does **NOT** exist

(B) is equal to 1

(C) is equal to 2

(D) is equal to 3

Ans. C

Sol. $f(x)=\sqrt{x}$

$\frac{1}{n+1}\le x<\frac{1}{n}$

$n<\frac{1}{x}\le n+1$

$n<\frac{1}{x}$

$n\ge\frac{1}{x}-1$

$n\ge\frac{1-x}{x}$

$\frac{1-x}{x}\le n<\frac{1}{x}$

$\sqrt{\frac{1-x}{x}}\le\sqrt{n}<\frac{1}{\sqrt{x}}$

$\sqrt{\frac{1-x}{x}}\le f(x)<\frac{1}{\sqrt{x}}$

$\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt\times\sqrt{\frac{1-x}{x}}\le f(x)g(x)<\frac{1}{\sqrt{x}}\times 2\sqrt{x}$

$\lim_{x\to 0}\int_{x^2}^{x}\sqrt{\frac{1-t}{t}}dt\sqrt{\frac{1-x}{x}}\le\lim_{x\to 0}f(x)g(x)<\lim_{x\to 0}2$

$\lim_{x\to 0}\frac{\int_{x^2}^{x}\left(\sqrt{\frac{1-t}{t}}\right)dt}{\sqrt{\frac{x}{1-x}}}\left(\frac{0}{0}\right)$

L.H. Rule

$$\lim_{x\to 0}\frac{\sqrt{\frac{1-x}{x}}-\sqrt{\frac{1-x^2}{x^2}}\times(2x)}{\frac{1}{2\sqrt{\frac{x}{1-x}}}\times\left(\frac{1-x+x}{(1-x)^2}\right)}=\frac{\sqrt{\frac{1-x}{x}}\left(1-\sqrt{\frac{1+x}{x}}\times 2x\right)}{\sqrt{\frac{1-x}{x}}\times\frac{1}{(1-x)^2}}=2$$

5. Let Q be the cube with the set of vertices $\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1,x_2,x_3\in\{0,1\}\}$. Let F be the set of all twelve lines containing the diagonals of the six faces of the cube Q. Let S be the set of all four lines containing the main diagonals of the cube Q; for instance, the line passing through the vertices (0,0,0) and (1,1,1) is in S. For lines ℓ_1 and ℓ_2, let $d(\ell_1,\ell_2)$ denote the shortest distance between them. Then the maximum value of $d(\ell_1,\ell_2)$, as ℓ_1 varies over F and ℓ_2 varies over S, is

(A) $\frac{1}{\sqrt{6}}$ (B) $\frac{1}{\sqrt{8}}$ (C) $\frac{1}{\sqrt{3}}$ (D) $\frac{1}{\sqrt{12}}$

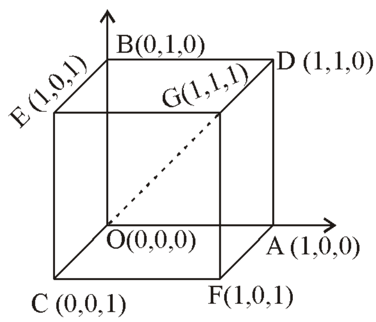
Ans. A

Sol. Let l_1 = Body diagonal

$\Rightarrow \vec{r}=(0,0,0)+\lambda(1,1,1)$

Let l_2 = FD

$\vec{r}=(1,1,0)+\mu(0,1,-1)$



$$d=\left|\frac{[\overrightarrow{AB}\ \vec{p}\ \vec{q}]}{(\vec{p}\times\vec{q})}\right|$$

$$\vec{p}\times\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 0 & 1 & -1\end{vmatrix}=-2\hat{i}+\hat{j}+\hat{k}$$

$$|\vec{p}\times\vec{q}|=\sqrt{6}$$

$$\overrightarrow{AB}=(1,1,0)$$

$$\overrightarrow{AB}\cdot(\vec{p}\times\vec{q})=(-2+1-0)=-1$$

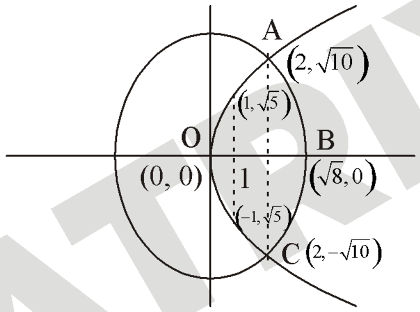
$$d=\left|\frac{-1}{\sqrt{6}}\right|=\frac{1}{\sqrt{6}}$$

6. Let $X = \left\{(x,y) \in Z \times Z : \frac{x^2}{8} + \frac{y^2}{20} < 1 \text{ and } y^2 < 5x\right\}$. Three distinct points P, Q and R are randomly chosen from X. Then the probability that P, Q and R form a triangle whose area is a positive integer, is

(A) $\frac{71}{220}$ (B) $\frac{73}{220}$ (C) $\frac{79}{220}$ (D) $\frac{83}{220}$

Ans. B

Sol. $X = \left\{(x,y) \in Z \times Z : \frac{x^2}{8} + \frac{y^2}{20} < 1 \ \& \ y^2 < 5x\right\}$



$\frac{x^2}{8} + \frac{y^2}{20} = 1$

$y^2 = 5x$

$\frac{x^2}{8} + \frac{5x}{20} = 1$

$5x^2 + 10x = 40$

$x^2 + 2x - 8 = 0$

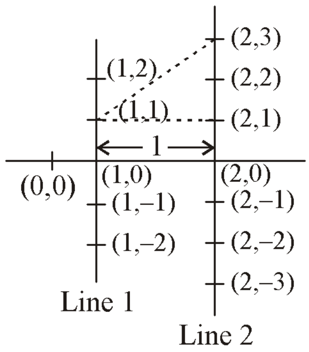
$x = -4, 2$

Integral points in OABC

(1,0), (1,1), (1, –1), (1,2), (1, –2)

(2,0), (2,1), (2,2), (2,3)

(2, –1), (2,–2), (2, –3)



$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height} = \text{integer}$

For any 3 points heights = 1

$\Delta = \frac{1}{2} \times \text{base} \times 1 = \text{integer}$

So base = even no.

From line 1 = 4 (no. of ways when distance between 2 points is 2) × 7 (No. of ways of selecting 3rd vertex from 2nd line)

From line 2 = (5 (when distance between 2 point is 2) + 3 (distance = 4) + 1 (distance = 6)) × 5 (No. of ways of selecting 3rd vertex from 1st line)

Total favourable cases = 28 + 9 × 5 = 73

$\text{Probability} = \frac{73}{^{12}C_3} = \frac{73}{220}$

7. Let P be a point on the parabola $y^2 = 4ax$, where $a > 0$. The normal to the parabola at P meets the x-axis at a point Q. The area of the triangle PFQ, where F is the focus of the parabola, is 120. If the slope m of the normal and a are both positive integers, then the pair (a, m) is

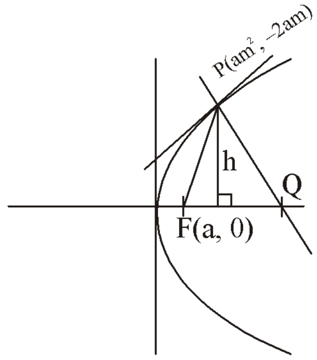
(A) (2,3) (B) (1,3) (C) (2,4) (D) (3,4)

Ans. A

Sol. Equation of normal

$y = mx - 2am - am^3$

$Q\ (2a + am^2, 0)$



$\Delta PFQ = \frac{1}{2} \times h \times FQ = 120$

$\frac{1}{2}|-2am|\left(a + am^2\right) = 120$

$a^2\,|m\,(1 + m^2)| = 120$

Now check options

$a = 2$ $m = 3$

SECTION 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER.**
- For each question, enter the correct integer corresponding to the answer using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

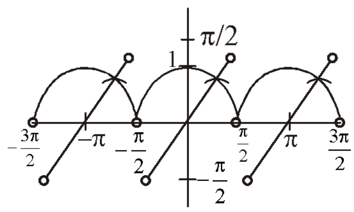
Full Marks : +4 If **ONLY** the correct integer is entered;

Zero Marks : 0 In all other cases.

8. Let $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, for $x \in R$. Then the number of real solutions of the equation $\sqrt{1+\cos(2x)} = \sqrt{2}\tan^{-1}(\tan x)$ in the set $\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is equal to

Ans. 3

Sol. $\sqrt{1+\cos 2x} = \sqrt{2}\tan^{-1}(\tan x)$



$\sqrt{2\cos^2 x} = \sqrt{2}\tan^{-1}(\tan x)$

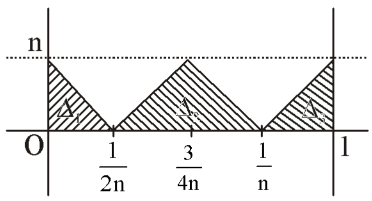
$\sqrt{2}|\cos x| = \sqrt{2}\tan^{-1}(\tan x)$

$y = |\cos x|$ $y = \tan^{-1}(\tan x)$

3 Solution

9. Let $n \ge 2$ be a natural number and $f:[0,1] \to R$ be the function defined by

$$f(x)=\begin{cases} n(1-2nx) & \text{if } 0 \le x \le \frac{1}{2n} \\ 2n(2nx-1) & \text{if } \frac{1}{2n} \le x \le \frac{3}{4n} \\ 4n(1-nx) & \text{if } \frac{3}{4n} \le x \le \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$



If n is such that the area of the region bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is 4, then the maximum value of the function f is

Ans. 8

Sol.
$$f(x)=\begin{cases} n(1-2nx) & 0 \le n \le \frac{1}{2n} \\ 2n(2nx-1) & \frac{1}{2n} \le x \le \frac{3}{4n} \\ 4n(1-nx) & \frac{3}{4n} \le x \le \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \frac{1}{n} \le x \le 1 \end{cases}$$

$\Delta_1 + \Delta_2 + \Delta_3 = 4$

$$\frac{1}{2}n \times \frac{1}{2n} + \frac{1}{2} \times n \times \frac{1}{2n} + \frac{1}{2} \times n \times \left(1 - \frac{1}{n}\right) = 4$$

$$4 = \frac{n}{2}$$

$n = 8$

10. Let $7\overbrace{5\cdots5}^{r}7$ denote the $(r+2)$ digit number where the first and the last digits are 7 and the remaining r digits are 5. Consider the sum $S = 77 + 757 + 7557 + \cdots + 7\overbrace{5\cdots5}^{98}7$. If $S = \frac{7\overbrace{5\cdots5}^{99}7 + m}{n}$, where m and n are natural numbers less than 3000, then the value of $m + n$ is

Ans. 1219

Sol. $S = 77 + 757 + 7557 + ... + 7\overbrace{5.....5}^{98}7$

$= 7\left(10 + 10^2 + ... + 10^{99}\right) + 50\left(1 + 11 + ... + \overbrace{111...1}^{98}\right) + 7 \times 99$

$= 70\left(\frac{10^{99} - 1}{9}\right) + \frac{50}{9}\left[(10-1) + \left(10^2 - 1\right) + ... + \left(10^{98} - 1\right)\right] + 7 \times 99$

$= 70\left(\frac{10^{99} - 1}{9}\right) + \frac{50}{9}\left[10\left(\frac{10^{98} - 1}{9}\right) - 98\right] + 7 \times 99$

$= \frac{7 \times 10^{100}}{9} - \frac{70}{9} + \frac{50}{9}\left[\frac{10^{99} - 1 - 9}{9} - 98\right] + 7 \times 99$

$= \frac{7 \times 10^{100}}{9} - \frac{70}{9} + \frac{50}{9}\left[\overbrace{111....1}^{99} - 99\right] + 7 \times 99$

$= \frac{7 \times 10^{100} - 70 + \overbrace{555...50}^{99}}{9} - 550 + 693$

$= \frac{7\overbrace{555...5}^{99} - 70 + 143 \times 9}{9}$

$= \frac{7\overbrace{55...5}^{99}7 + 1210}{9}$

$m + n = 1219$

11. Let $A = \left\{\frac{1967 + 1686i\sin\theta}{7 - 3i\cos\theta} : \theta \in R\right\}$. If A contains exactly one positive integer n, then the value of n is

Ans. 281

Sol. $A = \frac{1967 + 1686i\sin\theta}{7 - 3i\cos\theta}$

$= \frac{281(7 + 6i\sin\theta)}{7 - 3i\cos\theta} \times \frac{7 + 3i\cos\theta}{7 + 3i\cos\theta}$

$= \frac{281\left(49 - 18\sin\theta\cos\theta + i\left(21\cos\theta + 42\sin\theta\right)\right)}{49 + 9\cos^2\theta}$

for positive integer

Im(A) = 0

21 cos θ + 42 sin θ = 0

$\tan\theta = \frac{-1}{2}; \sin 2\theta = \frac{-4}{5}, \cos^2\theta = \frac{4}{5}$

$\text{Re}(A) = \frac{281(49 - 9\sin 2\theta)}{49 + 9\cos^2\theta}$

$= \frac{281\left(49 - 9 \times \frac{-4}{5}\right)}{49 + 9 \times \frac{4}{5}} = 281(+\text{ve integer})$

12. Let P be the plane $\sqrt{3}x + 2y + 3z = 16$ and let

$S = \left\{\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} : \alpha^2 + \beta^2 + \gamma^2 = 1 \text{ and the distance of } (\alpha, \beta, \gamma) \text{ from the plane P is } \frac{7}{2}\right\}$.

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be three distinct vectors in S such that $|\vec{u} - \vec{v}| = |\vec{v} - \vec{w}| = |\vec{w} - \vec{u}|$. Let V be the volume of the parallelopiped determined by vectors $\vec{u}, \vec{v}$ and $\vec{w}$. Then the value of $\frac{80}{\sqrt{3}}V$ is

Ans. 45

Sol. $P : \sqrt{3}x + 2y + 3z = 16$

$S = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k} : \alpha^2 + \beta^2 + \gamma^2 = 1$

$d(\alpha, \beta, \gamma)$ from $P = \frac{7}{2}$

$|\vec{u} - \vec{v}| = |\vec{v} - \vec{w}| = |\vec{w} - \vec{u}|$

V : volume of parallelepiped by vectors $\vec{u}, \vec{v}, \vec{w}$

$\frac{80}{\sqrt{3}}V = ?$

$d(\alpha, \beta, \gamma)$ from P= 7/2 (Given)

$\Rightarrow \frac{|\sqrt{3}\alpha + 2\beta + 3\gamma - 16|}{\sqrt{3 + 4 + 9}} = \frac{7}{2}$

$= \frac{|\sqrt{3}\alpha + 2\beta + 3\gamma - 16|}{4} = \frac{7}{2}$

$|\sqrt{3}\alpha + 2\beta + 3\gamma - 16| = 14$ __________(1)

$\alpha^2 + \beta^2 + \gamma^2 = 1$ __________(2)

Volume of parallelepiped by vector $\vec{u}, \vec{v}, \vec{w}$

$V = [\vec{u} \quad \vec{v} \quad \vec{w}]$

$= \vec{u}\cdot(\vec{v}\times\vec{w})$ ____________(3)

$|\vec{u}| = |\vec{v}| = |\vec{w}| = 1$ (Given) ____________(4)

$|\vec{u}-\vec{v}| = |\vec{v}-\vec{w}| = |\vec{w}-\vec{u}|$ (Given)

$\Rightarrow |\vec{u}-\vec{v}|^2 = |\vec{v}-\vec{w}|^2 = |\vec{w}-\vec{u}|^2$

$\Rightarrow u^2 + v^2 - 2\vec{u}\cdot\vec{v} = v^2 + w^2 - 2\vec{v}\cdot\vec{w}$

(A) (B)

$= w^2 + u^2 - 2\vec{w}\cdot\vec{u}$

(C)

(A) and (B)

$\Rightarrow u^2 + v^2 - 2\vec{u}\cdot\vec{v} = v^2 + w^2 - 2\vec{v}\cdot\vec{w}$

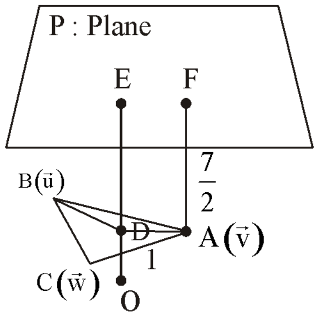
$\Rightarrow u^2 - w^2 = 2\vec{u}\cdot\vec{v} - 2\vec{v}\cdot\vec{w}$ $\left[|\vec{u}| = |\vec{w}| = 1 \ (\text{Given})\right]$

$\Rightarrow \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{w}$

Hence, by using (B) and (C) also, we will get

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{u} = m \ (\text{say})$ ...(5)

$\Rightarrow \vec{u}, \vec{v}, \vec{w}$ are the vectors of an equilateral triangle (say ΔABC)



$d(O, P) = \dfrac{16}{\sqrt{3+4+9}}$

$= \dfrac{16}{4}$

$= 4$ units

$\overrightarrow{OA} = \vec{u}, \overrightarrow{OB} = \vec{v}, \overrightarrow{OC} = \vec{w}$

$|\overrightarrow{OA}| = |\overrightarrow{OB}| = |\overrightarrow{OC}| = 1$ (Given)

In an equilateral triangle, circumcentre, orthocentre and centroid coincide.

Let D be the circumcentre of ΔABC, then

$\angle ADB = 120°$

Given $= \dfrac{DA^2 + DB^2 - AB^2}{2(DA)\cdot(DB)}$ ...(6)

OE = OD + DE

= OD + AF

$\Rightarrow 4 = OD + \dfrac{7}{2}$

$\Rightarrow OD = 4 - \frac{7}{2} = \frac{1}{2}$

$\Rightarrow DA = \sqrt{OA^2 - OD^2}$

$= \sqrt{1 - \frac{1}{4}}$

$DA = \frac{\sqrt{3}}{2}$

$\Rightarrow DA = DB = \frac{\sqrt{3}}{2}$ ________(7)

From (6) and (7),

$$-\frac{1}{2} = \frac{\frac{3}{4} + \frac{3}{4} - AB^2}{2\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}}$$

$$-\frac{1}{2} = \frac{\frac{3}{2} - AB^2}{\frac{3}{2}}$$

$\Rightarrow -\frac{1}{2} \times \frac{3}{2} = \frac{3}{2} - AB^2$

$\Rightarrow AB^2 = \frac{3}{2} + \frac{3}{4}$

$\Rightarrow AB^2 = \frac{9}{4}$

$\Rightarrow AB - \frac{3}{2} = |\vec{u} - \vec{v}|$

$\Rightarrow AB^2 = \frac{9}{4} = u^2 + v^2 - 2\vec{u} - \vec{v}$

$\Rightarrow \frac{9}{4} = 1 + 1 - 2m$

$\Rightarrow 2m = 2 - \frac{9}{4} = -\frac{1}{4}$

$\Rightarrow m = -\frac{1}{8}$ ________(8)

Volume of parallelepiped

$$V = \left|\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]\right|$$

$$\left|\vec{u}\ \ \vec{v}\ \ \vec{w}\right|^2 = \begin{vmatrix} 1 & \vec{u}\cdot\vec{v} & \vec{u}\cdot\vec{w} \\ \vec{u}\cdot\vec{v} & 1 & \vec{v}\cdot\vec{w} \\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{v} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & m & m \\ m & 1 & m \\ m & m & 1 \end{vmatrix}$$

$$= 1\left(1-m^2\right) - m\left(m-m^2\right) + m\left(m^2-m\right)$$

$= 1 - m^2 - m^2 + m^3 + m^3 - m^2$

$= 1 - 3m^2 + 2m^3$

$$\left|\vec{u}\ \ \vec{v}\ \ \vec{w}\right| = 2m^3 - 3m^2 + 1$$

$= (m - 1)\ [2m^2 - m - 1]$

$= (m - 1)\ [2m^2 - 2m + m - 1]$

$= (m - 1)\ (m - 1)\ (2m + 1)$

$= (m - 1)^2\ (2m + 1)$

$$\Rightarrow \left|\left[\vec{u}\ \ \vec{v}\ \ \vec{w}\right]\right| = (m-1)\sqrt{(2m+1)} = v$$

$$\left|\left(-\frac{1}{8}-1\right)\sqrt{2\times-\frac{1}{8}+1}\right|$$

$$V = \frac{9}{8}\times\frac{\sqrt{3}}{2}$$

$$\frac{80}{\sqrt{3}}v = \frac{80}{\sqrt{3}}\times\frac{9}{8}\times\frac{\sqrt{3}}{2}$$

$= 45$

13. Let a and b be two nonzero real numbers. If the coefficient of x^5 in the expansion of $\left(ax^2 + \frac{70}{27bx}\right)^4$ is equal to the coefficient of x^{-5} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^7$, then the value of 2b is

Ans. 3

Sol. $T_{r+1} = {}^4C_r\left(a\cdot x^2\right)^{4-r}\cdot\left(\frac{70}{27bx}\right)^r$

$= {}^4C_r\cdot a^{4-r}\cdot\frac{70^r}{(27b)^r}\cdot x^{8-3r}$

here $8 - 3r = 5$

$8 - 5 = 3r \Rightarrow r = 1$

$\therefore$ coeff. $= 4\cdot a^3\cdot\frac{70}{27b}$

$T_{r+1} = {}^7C_r(ax)^{7-r}\left(\frac{-1}{bx^2}\right)^r$

$= {}^7C_r\cdot a^{7-r}\left(\frac{-1}{b}\right)^r\cdot x^{7-3r}$

$7 - 3r = -5 \Rightarrow 12 = 3r \Rightarrow r = 4$

Coeff. : ${}^7C_4\cdot a^3\cdot\left(\frac{-1}{b}\right)^4 = \frac{35a^3}{b^4}$

Now $\frac{35a^3}{b^4} = \frac{280a^3}{27b}$

$b^3 = \frac{35\times 27}{280} = b = \frac{3}{2} \Rightarrow 2b = 3$

SECTION 4 (Maximum Marks: 12)

- This section contains **FOUR (04)** Matching List Sets.
- Each set has **ONE** Multiple Choice Question.
- Each set has **TWO** lists: **List-I** and **List-II**.
- **List-I** has Four entries (P), (Q), (R) and (S) and **List-II** has Five entries (1), (2), (3), (4) and (5).
- **FOUR** options are given in each Multiple Choice Question based on **List-I** and **List-II** and **ONLY ONE** of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 **ONLY** if the option corresponding to the correct combination is chosen;
Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
Negative Marks : –1 In all other cases.

14. Let α, β and γ be real numbers. Consider the following system of linear equations

$x + 2y + z = 7$

$x + \alpha z = 11$

$2x - 3y + \beta z = \gamma$

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	List-II
(P) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma = 28$, then the system has	(1) a unique solution
(Q) If $\beta = \frac{1}{2}(7\alpha - 3)$ and $\gamma \neq 28$, then the system has	(2) no solution
(R) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma \neq 28$, then the system has	(3) infinitely many solutions
(S) If $\beta \neq \frac{1}{2}(7\alpha - 3)$ where $\alpha = 1$ and $\gamma = 28$, then the system has	(4) $x = 11$, $y = -2$ and $z = 0$ as a solution
	(5) $x = -15$, $y = 4$ and $z = 0$ as a solution

The correct option is:
(A) (P) $\rightarrow$ (3); (Q) $\rightarrow$ (2); (R) $\rightarrow$ (1); (S) $\rightarrow$ (4)
(B) (P) $\rightarrow$ (3); (Q) $\rightarrow$ (2); (R) $\rightarrow$ (5); (S) $\rightarrow$ (4)
(C) (P) $\rightarrow$ (2); (Q) $\rightarrow$ (1); (R) $\rightarrow$ (4); (S) $\rightarrow$ (5)
(D) (P) $\rightarrow$ (2); (Q) $\rightarrow$ (1); (R) $\rightarrow$ (1); (S) $\rightarrow$ (3)

Sol. $D = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & \alpha \\ 2 & -3 & \beta \end{vmatrix}$

$= 1\,[3\alpha] - 2\,[\beta - 2\alpha] + [-3]$
$= 3\alpha - 2\beta + 4\alpha - 3$
$= 7\alpha - 2\beta - 3$
If $D = 0$
$7\alpha - 2\beta - 3 = 0$

$$\beta = \frac{7\alpha - 3}{2}$$

$$D_x = \begin{vmatrix} 7 & 2 & 1 \\ 11 & 0 & \alpha \\ \gamma & -3 & \beta \end{vmatrix}$$

if $\gamma = 28$ & $\beta = \frac{7\alpha - 3}{2}$

$$\text{then } D_x = \begin{vmatrix} 7 & 2 & 1 \\ 11 & 0 & \alpha \\ 28 & -3 & \frac{7\alpha - 3}{2} \end{vmatrix}$$

$$7[3\alpha]-2\left[11\frac{(7\alpha-3)}{2}-28\alpha\right]+[-33]$$

$21\alpha - [11\,(7\alpha - 3) - 56\alpha] - 33$

$21\alpha - (77\alpha - 33 - 56\alpha) - 33$

$= 0$

$$D_y = \begin{vmatrix} 1 & 7 & 1 \\ 1 & 11 & \alpha \\ 2 & \gamma & \beta \end{vmatrix}$$

Again if $\gamma = 28$ & $\beta = \dfrac{7\alpha-3}{2}$

$$D_y = \begin{vmatrix} 1 & 7 & 1 \\ 1 & 11 & \alpha \\ 2 & 28 & \frac{7\alpha-3}{2} \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$$= \begin{vmatrix} 0 & -4 & 1-\alpha \\ 1 & 11 & \alpha \\ 2 & 28 & \frac{7\alpha-3}{2} \end{vmatrix}$$

$$4\left[\frac{7\alpha-3}{2}-2\alpha\right]+(1-\alpha)(28-22)$$

$2\,(7\alpha - 3) - 8\alpha + 6\,(1 - \alpha)$

$14\alpha - 6 - 8\alpha + 6 - 6\alpha$

$= 0$

$$D_z = \begin{vmatrix} 1 & 2 & 7 \\ 1 & 0 & 11 \\ 2 & -3 & \gamma \end{vmatrix}$$

$1\,(33) - 2\,(\gamma - 22) + 7\,(-3)$

$33 + 44 - 21 - 2\gamma$

$\Rightarrow 56 - 2\gamma$

If $\gamma = 28$

$D_z = 0$

(P) If $\beta = \frac{1}{2}(7\alpha - 3)$ & $\alpha = 28$

then D_x, D_y, D_z are all zero

$\therefore$ Infinite many solution

$P \rightarrow (3)$

(Q) If $\beta = \frac{1}{2}(7\alpha - 3)$ & $\gamma \neq 28$

It can be easily observed that $D_z \neq 0$

$\therefore$ $D = 0$ & $D_z \neq 0$

$\Rightarrow$ No solution

$Q \rightarrow 2$

(R) If $\beta \neq \frac{1}{2}(7\alpha - 3)$, i.e. $D \neq 0$

It will be the case of unique solution but here additional info of $\alpha = 1$ & $\gamma \neq 28$ is provided

(S) If $\alpha = 1$ then

$x + 2y + z = 7$

$x + z = 11$

$2y = 7 - 11$

$y = -2$ will always be the solution.

But $\gamma \neq 28$ then $z \neq 0$

So, $R \rightarrow (1)$

Again

If $\alpha = 1$; $\gamma = 28$

then $y = -2$; $z = 0$

& $z + 2(-2) = 7$

$x = 7 + 4$

$x = 11$

So we get the solution $S \rightarrow (4)$

15. Consider the given data with frequency distribution

x_i	3	8	11	10	5	4
f_i	5	2	3	2	4	4

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	**List-II**
(P) The mean of the above data is	(1) 2.5
(Q) The median of the above data is	(2) 5
(R) The mean deviation about the mean of the above data is	(3) 6
(S) The mean deviation about the median of the above data is	(4) 2.7
	(5) 2.4

The correct option is:

(A) (P) $\rightarrow$ (3); (Q) $\rightarrow$ (2); (R) $\rightarrow$ (4); (S) $\rightarrow$ (5)

(B) (P) $\rightarrow$ (3); (Q) $\rightarrow$ (2); (R) $\rightarrow$ (1); (S) $\rightarrow$ (5)

(C) (P) $\rightarrow$ (2); (Q) $\rightarrow$ (3); (R) $\rightarrow$ (4); (S) $\rightarrow$ (1)

(D) (P) $\rightarrow$ (3); (Q) $\rightarrow$ (3); (R) $\rightarrow$ (5); (S) $\rightarrow$ (5)

Sol. Arrange the table in ascending order of x_i.

x_i	f_i	$f_i x_i$	$\lvert \bar{x} - x_i \rvert$	$\lvert \bar{x} - x_i \rvert f$
3	5	15	3	15
4	4	16	2	8
5	4	20	1	4
8	2	16	$\lvert -2 \rvert \Rightarrow 2$	4
10	2	20	$\lvert -4 \rvert \Rightarrow 4$	8
11	3	33	$\lvert -5 \rvert \Rightarrow 5$	15
	20	120		54

$$\bar{x} = \frac{120}{20} = 6$$

By observation 10, 11th term will be 5

$P \to (3)$

$\therefore\ Q \to 2$

(R) Mean deviation about mean $= \frac{54}{20} = 2.7$

$R \to 4$

(A)

16. Let ℓ_1 and ℓ_2 be the lines $\vec{r}_1 = \lambda(\hat{i} + \hat{j} + \hat{k})$ and $\vec{r}_2 = (\hat{j} - \hat{k}) + \mu(\hat{i} + \hat{k})$, respectively. Let X be the set of all the planes H that contain the line ℓ_1. For a plane H, let $d(H)$ denote the smallest possible distance between the points of ℓ_2 and H. Let H_0 be a plane in X for which $d(H_0)$ is the maximum value of $d(H)$ as H varies over all planes in X.

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	**List-II**
(P) The value of $d(H_0)$ is	(1) $\sqrt{3}$
(Q) The distance of the point (0,1,2) from H_0 is	(2) $\frac{1}{\sqrt{3}}$
(R) The distance of origin from H_0 is	(3) 0
(S) The distance of origin from the point of intersection of planes y = z, x = 1 and H_0 is	(4) $\sqrt{2}$
	(5) $\frac{1}{\sqrt{2}}$

(A) (P) $\to$ (2); (Q) $\to$ (4); (R) $\to$ (5); (S) $\to$ (1)
(B) (P) $\to$ (5); (Q) $\to$ (4); (R) $\to$ (3); (S) $\to$ (1)
(C) (P) $\to$ (2); (Q) $\to$ (1); (R) $\to$ (3); (S) $\to$ (2)
(D) (P) $\to$ (5); (Q) $\to$ (1); (R) $\to$ (4); (S) $\to$ (2)

Sol. $\overline{r_1} = \lambda(i+j+k)$; $\overline{r_2} = (j-k)+\mu(i+k)$

H_0 will be plane containing l_1 & parallel to l_2.

$\therefore$ ⊥r vector of plane parallel to lines l_1 & l_2 is $\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$

$\Rightarrow i\,[1] - j\,[1-1] + k\,[-1]$

$\Rightarrow i - k$

$\therefore$ Plane will be $x + 0\cdot y - z = \lambda$; passes through origin

$\therefore$ $\lambda = 0$

$\therefore$ Plane $\Rightarrow x - z = 0 \to H_0$ Plane

(P) $d(H_0) = 1$ distance of point $(0, 1, -1)$ from H

$d = \dfrac{|0-(-1)|}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $P \to 5$

(Q) $d = \left|\dfrac{0-2}{\sqrt{2}}\right| = \sqrt{2}$ $Q \to 4$

(R) $d = \left|\dfrac{0}{\sqrt{2}}\right| = 0$ $R \to 3$

(S) Point of intersection will be $(1,1,1)$

$d = \sqrt{1+1+1} = \sqrt{3}$

(B)

17. Let z be a complex number satisfying $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$, where $\bar{z}$ denotes the complex conjugate of z. Let the imaginary part of z be non-zero.

Match each entry in **List-I** to the correct entries in **List-II**.

List-I	**List-II**
(P) $\|z\|^2$ is equal to	(1) 12
(Q) $\|z-\bar{z}\|^2$ is equal to	(2) 4
(R) $\|z\|^2 + \|z+\bar{z}\|^2$ is equal to	(3) 8
(S) $\|z+1\|^2$ is equal to	(4) 10
	(5) 7

The correct option is :

(A) (P) $\to$ (1); (Q) $\to$ (3); (R) $\to$ (5); (S) $\to$ (4)

(B) (P) $\to$ (2); (Q) $\to$ (1); (R) $\to$ (3); (S) $\to$ (5)

(C) (P) $\to$ (2); (Q) $\to$ (4); (R) $\to$ (5); (S) $\to$ (1)

(D) (P) $\to$ (2); (Q) $\to$ (3); (R) $\to$ (5); (S) $\to$ (4)

Sol. $|z|^3 + 2z^2 + 4\bar{z} - 8 = 0$ __________(1)

taking conjugate

$|z|^3 + 2\bar{z}^2 + 4z - 8 = 0$ __________(2)

(1) – (2)

$2\left(z^2 - \bar{z}^2\right) + 4\left(\bar{z} - z\right) = 0$

$\left(z - \bar{z}\right)\left(z + \bar{z}\right) + 2\left(\bar{z} - z\right) = 0$

$\left(z - \bar{z}\right)\left(z + \bar{z} - 2\right) = 0$

$\therefore z - \bar{z} \neq 0$ (Otherwise Img part of z will be zero)

$\therefore z + \bar{z} - 2 = 0$

$x + iy + x - iy - 2 = 0$

$2x - 2 = 0$

$x = 1$

$\therefore$ z will be of the form $z = 1 + iy$

$|z| = \sqrt{1+y^2}$; $z^2 = 1 - y^2 + 2iy$; $\bar{z} = 1 - iy$

Substitute in equation (1)

$\left(1+y^2\right)^{\frac{3}{2}} + 2\left(1 - y^2 + 2iy\right) + 4\left(1 - iy\right) - 8 = 0$

$\left(1+y^2\right)^{\frac{3}{2}} + 2 - 2y^2 + 4 - 8 + i\left(4y - 4y\right) = 0$

$\left(1+y^2\right)^{\frac{3}{2}} + \left(-2y^2 - 2\right) = 0$

$\left(1+y^2\right)^{\frac{3}{2}} - 2\left(1+y^2\right) = 0$

$\left(1+y^2\right)\left\{\sqrt{1+y^2} - 2\right\} = 0$

$1 + y^2 = 4$

$y^2 = 3$

(P) $|z|^2 = 1 + y^2$

$= 4$

(Q) $\left|z-\bar{z}^{2}\right| \Rightarrow \left|(1+iy)-(1-iy)\right|^{2}$

$= |1 + iy - 1 + iy|^2$

$= |2iy|^2$

$= 4y^2$

$= 4 \cdot 3$

$= 12$

(R) $|z|^2 + |z+\bar{z}|^2$

$= 4 + |1 + iy + 1 - iy|^2$

$\Rightarrow 4 + 4$

$\Rightarrow 8$

(S) $|z + 1|^2 = |1 + iy + 1|^2$

$= |2 + iy|^2$

$= \left(\sqrt{4+y^2}\right)^2$

$= 4 + 3$

$= 7$