JEE Adv. 2022

Question Paper With Text Solution

28 August. | Paper-2

MATHS

JEE ADV. AUGUST 2022 | 28$^{TH.}$ AUGUST PAPER-2

SECTION 1 (Maximum marks: 24)

- This section contains EIGHT (08) questions.
- The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 TO 9, BOTH INCLUSIVE.
- For each question, enter the correct integer corresponding to the answer using the mouse and the onscreen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct integer is entered;
  Zero Marks : 0 If the question is unanswered;
  Negative Marks : -1 In all other cases.

1. Let α and β be real numbers such that $-\frac{\pi}{4}<\beta<0<\alpha<\frac{\pi}{4}$. If $\sin(\alpha+\beta)=\frac{1}{3}$ and $\cos(\alpha-\beta)=\frac{2}{3}$, then the greatest integer less than or equal to $\left(\frac{\sin\alpha}{\cos\beta}+\frac{\cos\beta}{\sin\alpha}+\frac{\cos\alpha}{\sin\beta}+\frac{\sin\beta}{\cos\alpha}\right)^2$ is ______________.

Ans. (1)

Sol. $\left(\frac{\sin\alpha}{\cos\beta}+\frac{\cos\alpha}{\sin\beta}+\frac{\cos\beta}{\sin\alpha}+\frac{\sin\beta}{\cos\alpha}\right)^2$

$$=\left(\frac{\cos(\alpha-\beta)}{\sin\beta\cos\beta}+\frac{\cos(\alpha-\beta)}{\sin\alpha\cdot\cos\alpha}\right)^2$$

$$=\left(\frac{4}{3}\left\{\frac{1}{\sin 2\beta}+\frac{1}{\sin 2\alpha}\right\}\right)^2$$

$$=\frac{16}{9}\left(\frac{2\sin(\alpha+\beta)\cdot\cos(\alpha-\beta)}{\sin 2\alpha\cdot\sin 2\beta}\right)^2$$

$$=\frac{16}{9}\left(\frac{4\cdot\frac{1}{3}\cdot\frac{2}{3}}{\cos(2\alpha-2\beta)-\cos(2\alpha+2\beta)}\right)^2$$

$$=\frac{16}{9}\left(\frac{\frac{8}{9}}{2\cos^2(\alpha-\beta)-1-1+2\sin^2(\alpha+\beta)}\right)^2$$

$$=\frac{16}{9}\left(\frac{\frac{8}{9}}{\frac{8}{9}-2+\frac{2}{9}}\right)^2 \qquad =\frac{16}{9}$$

2. If y(x) is the solution of the differential equation

$xdy - (y^2 - 4y)dx = 0$ for $x > 0$, $y(1) = 2$,

and the slope of the curve $y = y(x)$ is never zero, then the value of $10\,y(\sqrt{2})$ is ____________.

Ans. (8)

Sol. $xdy - \left(y^2 - 4y\right)dx = 0, x > 0$

$\int \frac{dy}{y^2 - 4y} = \int \frac{dx}{x}$

$\int \left(\frac{1}{y-4} - \frac{1}{y}\right)dy = 4\int \frac{dx}{x}$

$\log_e |y-4| - \log_e |y| = 4\log_e x + \log_e c$

$\frac{|y-4|}{|y|} = cx^4 \xrightarrow{(1,2)} c = 1$

$|y-4| = |y|\,x^4$

C–1 and C–2

C–1: $y - 4 = yx^4$

$y = \frac{4}{1-x^4}$

y(1)=ND (rejected)

C–2: $y - 4 = -yx^4$

$y = \frac{4}{1+x^4}$

$y(1) = 2$

$y(\sqrt{2}) = \frac{4}{5} \Rightarrow 10y(\sqrt{2}) = 8$

3. The greatest integer less than or equal to $\int_1^2 \log_2\left(x^3+1\right)dx + \int_1^{\log_2 9}\left(2^x - 1\right)^{\frac{1}{3}}dx$ is _______.

Ans. (5)

Sol. $y = \int_1^2 \log_2\left(x^3+1\right)dx + \int_1^{2\log_2 3}\left(2^x - 1\right)^{\frac{1}{3}}dx$

$= 4\log_2 3 - 1$

we know $\int_a^b f(x)dx + \int_{f(a)}^{f(b)}\left(f^{-1}(x)\right)dx = bf(b) - af(a)$

So, [y] = 5

4. The product of all positive real values of x satisfying the equation $x^{\left(16(\log_5 x)^3 - 68\log_5 x\right)} = 5^{-16}$ is____

Ans. (1)

Sol. Taking log to the base 5 on both sides

$$\left(16(\log_5 x)^3 - 68(\log_5 x)\right)(\log_5 x) = -16$$

Let $(\log_5 x) = t$

$16t^4 - 68t^2 + 16 = 0$

OR $4t^4 - 16t^2 - t^2 + 4 = 0$

OR $(4t^2 - 1)(t^2 - 4) = 0$

OR $t = \pm\frac{1}{2}, \pm 2$

So $\log_5 x = \pm\frac{1}{2}$ OR ± 2

$\Rightarrow x = 5^{\frac{1}{2}}, 5^{\frac{-1}{2}}, 5^2, 5^{-2}$

5. If $\beta = \lim_{x\to 0} \frac{e^{x^3} - (1-x^3)^{\frac{1}{3}} + \left((1-x^2)^{\frac{1}{2}} - 1\right)\sin x}{x\sin^2 x}$ then the value of 6β is ____.

Ans. (5)

Sol. $$\beta = \lim_{x\to 0} \frac{e^{x^3} - (1-x^3)^{1/3}}{\frac{x\sin^2 x}{x^2} x^2} + \frac{\left((1-x^2)^{1/2} - 1\right)\sin x}{x\frac{\sin^2 x}{x^2} x^2}$$

use expansion

$$\beta = \lim_{x\to 0} \frac{(1+x^3) - \left(1 - \frac{x^3}{3}\right)}{x^3} + \lim_{x\to 0} \frac{\left(\left(1 - \frac{x^2}{2}\right) - 1\right)}{x^2} \frac{\sin x}{x}$$

$$\beta = \lim_{x\to 0} \frac{4x^3}{3x^3} + \lim_{x\to 0} \frac{-x^2}{2x^2}$$

$$\beta = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$$

$6\beta = 5$

6. Let β be a real number. Consider the matrix $A=\begin{pmatrix}\beta & 0 & 1\\ 2 & 1 & -2\\ 3 & 1 & -2\end{pmatrix}$. If $A^7-(\beta-1)A^6-\beta A^5$ is a singular matrix, then the value of 9β is ______.

Ans. (3)

Sol. $A^7-(\beta-1)A^6-\beta A^5$ is a singular matrix, so

$|A^7-(\beta-1)A^6-\beta A^5|=0$

$|A|^5\,|(A^2-(\beta-1)A-\beta I)|=0$

$|A|^5\,|A^2-(\beta-1)A-\beta I|=0$

$|A|^5\,|A^2-\beta A+A-\beta I)|=0$

$|A|^5\,|(A+I)(A-\beta I)|=0$

here $|A|=-1$

$$|A+I|=\begin{bmatrix}\beta+1 & 0 & 1\\ 2 & 2 & -2\\ 3 & 1 & -1\end{bmatrix}=0+1(2-6)=-4$$

$$\text{So } |A-\beta I|=\begin{bmatrix}0 & 0 & 1\\ 2 & 1-\beta & -2\\ 3 & 1 & -2-\beta\end{bmatrix}$$

$\Rightarrow 2-3+3\beta=0$

$\Rightarrow 3\beta=1$

So $9\beta=3$

7. Consider the hyperbola $\frac{x^2}{100}-\frac{y^2}{64}=1$ with foci at S and S_1, where S lies on the positive x-axis. Let P be a point on the hyperbola, in the first quadrant. Let $\angle SPS_1=\alpha$, with $\alpha<\frac{\pi}{2}$. The straight line passing through the point S and having the same slope as that of the tangent at P to the hyperbola, intersects the straight line S_1P at P_1. Let δ be the distance of P from the straight line SP_1, and $\beta=S_1P$. Then the greatest integer less than or equal to $\frac{\beta\delta}{9}\sin\frac{\alpha}{2}$ is ____.

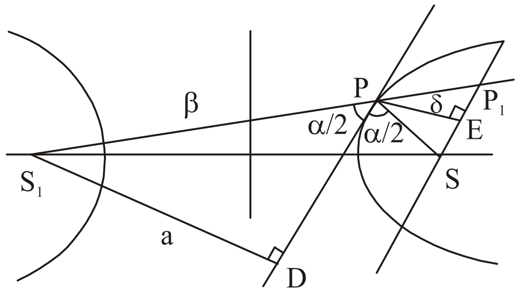
Ans. (7)

Sol. Let $S_1D=a$ in S_1PD $\qquad \sin\left(\frac{\alpha}{2}\right)=\frac{a}{\beta}$

Now $\frac{1}{9}\beta\delta\sin\left(\frac{\alpha}{2}\right)=\frac{1}{9}\beta\delta\cdot\frac{a}{\beta}=\frac{a\delta}{9}=\frac{b^2}{9}=\frac{64}{9}$

we know product of perpendiculars from foci upon any tangent to Hyperbola = b^2, $(a\delta = b^2)$

Hence $\left[\frac{64}{9}\right] = 7$



8. Consider the functions $f, g : R \to R$ defined by $f(x) = x^2 + \frac{5}{12}$ and $g(x) = \begin{cases} 2\left(1 - \frac{4|x|}{3}\right) & |x| \le \frac{3}{4}, \\ 0, & |x| > \frac{3}{4} \end{cases}$.

If α is the area of the region $\left\{(x, y) \in R \times R : |x| \le \frac{3}{4}, 0 \le y \le \min\{f(x), g(x)\}\right\}$ then the value of

9α is ____.

Ans. (6)

Sol. $f(x) = x^2 + \frac{5}{12}$

$g(x) = 2\left(1 - \frac{4}{3}|x|\right)$

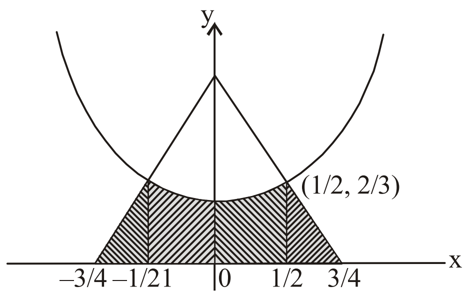
$|x| \le \frac{3}{4}$ and $0 \le y \le (f(x), g(x))$

for $x \ge 0$

$x^2 + \frac{5}{12} = 2 - \frac{8}{3}x$

$\Rightarrow 12x^2 + 32x - 19 = 0$

$12x^2 + 38x - 6x - 19 = 0 \Rightarrow x = \frac{1}{2}$



Required area $= 2\left(\int_0^{1/2}\left(x^2 + \frac{5}{12}\right)dx + \frac{1}{2} \times \frac{1}{4} \times \left(2 - \frac{4}{3}\right)\right)$

$\Rightarrow \alpha = 2\left(\frac{1}{24} + \frac{5}{24} + \frac{2}{24}\right) = \frac{2}{3}$ sq.units $\qquad \Rightarrow 9\alpha = 9 \times \frac{2}{3} = 6$

SECTION 2 (Maximum marks: 24)

- This section contains SIX (06) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 ONLY if (all) the correct option(s) is(are) chosen;

  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

  Zero Marks : 0 If unanswered;

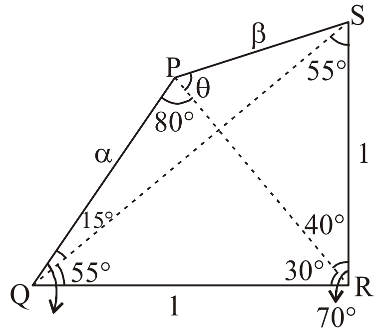
  Negative Marks: –2 In all other cases.

9. Let PQRS be a quadrilateral in plane, where QR = 1, $\angle PQR = \angle QRS = 70^\circ$, $\angle PQS = 15^\circ$ and $\angle PRS = 40^\circ$. If $\angle RPS = \theta^\circ$, $PQ = \alpha$ and $PS = \beta$, then interval(s) that contain(s) the value of $4\alpha\beta \sin\theta^\circ$ is /are :

(A) $\left(0,\sqrt{2}\right)$ (B) $(1, 2)$ (C) $\left(\sqrt{2},3\right)$ (D) $\left(2\sqrt{2},3\sqrt{2}\right)$

Ans. (A, B)

Sol. Figure as shown, can be drawn on the basis of given data.



Applying sine rule in ΔPQR,

$$\frac{\alpha}{\sin 30^\circ} = \frac{1}{\sin 80^\circ}$$

$$\Rightarrow \alpha = \frac{1}{2\sin 80^\circ} \qquad \text{\_\_\_\_\_\_\_\_(i)}$$

Applying sine rule in ΔPRS,

$$\frac{\beta}{\sin 40^\circ} = \frac{1}{\sin\theta}$$

$\Rightarrow \beta\sin\theta = \sin 40^\circ$ ________(ii)

From (i) & (ii),

$$4\alpha\beta\sin\theta = 4\cdot\frac{1}{2\sin 80^\circ}\cdot\sin 40^\circ = \frac{1}{\cos 40^\circ}$$

Using $\cos 0^\circ = 1, \cos 30^\circ = \frac{\sqrt{3}}{2}, \cos 45^\circ = \frac{1}{\sqrt{2}}$ and $\cos 60^\circ = \frac{1}{2}$

Only Options (A) and (B) are correct

10. Let $\alpha = \sum_{k=1}^{\infty} \sin^{2k}\left(\frac{\pi}{6}\right)$. Let $g:[0,1]\to R$ be the function defined by $g(x) = 2^{\alpha x} + 2^{\alpha(1-x)}$.

Then, which of the following statements is/are TRUE ?

(A) The minimum value of g(x) is $2^{7/6}$

(B) The maximum value of g(x) is $1 + 2^{1/3}$

(C) The function g(x) attains its maximum at more than one point

(D) The function g(x) attains its minimum at more than one point

Ans. (A, B, C)

Sol. $$\alpha = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{2k} = \sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^{k} = \frac{1/4}{1-\frac{1}{4}} = \frac{1}{3}$$

Hence, $g(x) = 2^{x/3} + 2^{(1-x)/3}$

Now, $$g'(x) = \frac{\ln 2}{3}\frac{\left(2^{2x/3} - 2^{1/3}\right)}{2^{x/3}}$$

$g'(x) = 0$ at $x = \frac{1}{2}$

And, derivative changes sign from negative to positive at $x = \frac{1}{2}$, hence $x = \frac{1}{2}$ is point of local minimum as well as absolute minimum of g(x) for $x \in [0, 1]$

Hence, minimum value of $g(x) = g\left(\frac{1}{2}\right)$

$= 2^{1/6} + 2^{1/6} = 2^{7/6}$

$\Rightarrow$ Option (A) is correct

Maximum value of g(x) is either equal to g(0) or g(1).

$g(0) = 1 + 2^{1/3}$
$g(1) = 2^{1/3} + 1$
Hence (B) and (C) are also correct

11. Let $\overline{z}$ denote the complex conjugate of a complex number z. If z is a non-zero complex number for which both real and imaginary parts of $\left(\overline{z}\right)^2 + \frac{1}{z^2}$ are integers, then which of the following is/are possible value(s) of $|z|$?

(A) $\left(\frac{43+3\sqrt{205}}{2}\right)^{\frac{1}{4}}$ (B) $\left(\frac{7+\sqrt{33}}{4}\right)^{\frac{1}{4}}$ (C) $\left(\frac{9+\sqrt{65}}{4}\right)^{\frac{1}{4}}$ (D) $\left(\frac{7+\sqrt{13}}{6}\right)^{\frac{1}{4}}$

Ans. (A)

Sol. Let $z = r(\cos\theta + i\sin\theta)$

$$\overline{z}^2 + \frac{1}{z^2} = r^2(\cos 2\theta - i\sin 2\theta)$$

$$+\frac{1}{r^2}(\cos 2\theta - i\sin 2\theta)$$

$$= \left(r^2 + \frac{1}{r^2}\right)(\cos 2\theta - i\sin 2\theta)$$

How

$$\left.\begin{aligned}\left(r^2+\frac{1}{r^2}\right)\cos 2\theta = \lambda \Rightarrow \left(r^4+\frac{1}{r^4}+2\right)\cos^2 2\theta = \lambda^2 \\ \left(r^2+\frac{1}{r^2}\right)\sin 2\theta = \mu \Rightarrow \left(r^4+\frac{1}{r^4}+2\right)\sin^2 2\theta = \mu^2\end{aligned}\right\} \Rightarrow r^4+\frac{1}{r^4}+2 = \lambda^2+\mu^2$$

So, $r^4 + \frac{1}{r^4} \in I$

(A) $r^4 = \frac{43+3\sqrt{205}}{2} \Rightarrow \frac{1}{r^4} = \frac{2(43-3\sqrt{205})}{1849-1845} = \frac{43-3\sqrt{205}}{2} \Rightarrow r^4 + \frac{1}{r^4} = 43 \in I$

(B) $r^4 = \frac{7+\sqrt{33}}{4} \Rightarrow \frac{1}{r^4} = \frac{4(7-\sqrt{33})}{49-33} = \frac{7-\sqrt{33}}{4} \Rightarrow r^4 + \frac{1}{r^4} \notin I$

(C) $r^4 = \frac{9+\sqrt{65}}{4} \Rightarrow \frac{1}{r^4} = \frac{4(9-\sqrt{65})}{81-65} = \frac{9-\sqrt{65}}{4} \Rightarrow r^4 + \frac{1}{r^4} \notin I$

(D) $r^4 = \frac{7+\sqrt{13}}{6} \Rightarrow \frac{1}{r^4} = \frac{6(7-\sqrt{13})}{49-13} = \frac{6(7-\sqrt{13})}{36} = \frac{7-\sqrt{13}}{6} \Rightarrow r^4 + \frac{1}{r^4} \notin I$

12. Let G be a circle of radius R > 0. Let $G_1, G_2, \ldots, G_n$ be n circles of equal radius r > 0. Suppose each of the n circles $G_1, G_2, \ldots, G_n$ touches the circle G externally. Also, for i = 1, 2, … , n – 1, the circle G_i touches G_{i+1} externally, and G_n touches G_1 externally. Then, which of the following statements is/are TRUE ?

(A) If n = 4, then $(\sqrt{2}-1)r < R$
(B) If n = 5, then $r < R$
(C) If n = 8, then $(\sqrt{2}-1)r < R$
(D) If n = 12, then $\sqrt{2}(\sqrt{3}+1)r > R$

Ans. (C, D)

Sol. $2(R+r)\sin\frac{\pi}{n} = 2r$

$\frac{R+r}{r} = \operatorname{cosec}\frac{\pi}{n}$

(A) $n = 4, R + r = \sqrt{2}r$

(B) $n = 5, \frac{R+r}{r} = \operatorname{cosec}\frac{\pi}{5} < \operatorname{cosec}\frac{\pi}{6}$

$R + r < 2r \Rightarrow r > R$

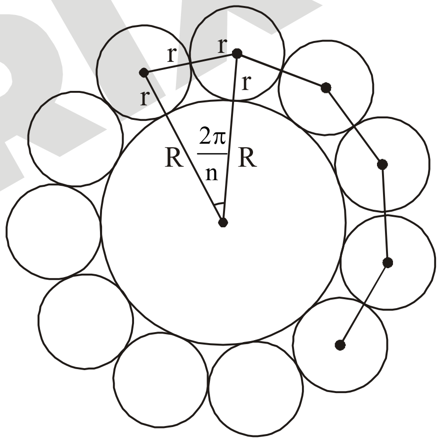
(C) $n = 8, \frac{R+r}{r} = \operatorname{cosec}\frac{\pi}{8} > \operatorname{cosec}\frac{\pi}{4}$

$R + r > \sqrt{2}r$

(D) $n = 12, \frac{R+r}{r} = \operatorname{cosec}\frac{\pi}{12} = \sqrt{2}(\sqrt{3}+1)$

$R + r = \sqrt{2}(\sqrt{3}+1)r$

$\sqrt{2}(\sqrt{3}+1)r > R$



13. Let $\hat{i}, \hat{j}$ and $\hat{k}$ be the unit vectors along the three positive coordinate axes.

Let
$$\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$$
$$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}, \quad b_2, b_3 \in R \quad ,$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}, \quad c_1, c_2, c_3 \in R$$

be three vectors such that $b_2b_3 > 0, \vec{a}.\vec{b} = 0$ and $\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3-c_1 \\ 1-c_2 \\ -1-c_3 \end{pmatrix}$. Then, which of the following is /are TRUE >

(A) $\vec{a} \cdot \vec{c} = 0$
(B) $\vec{b} \cdot \vec{c} = 0$
(C) $|\vec{b}| > \sqrt{10}$
(D) $|\vec{c}| \le \sqrt{11}$

Ans. (B, C, D)

Sol. $\vec{a} = 3\hat{i} + \hat{j} - \hat{k}$

$\vec{b} = \hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\begin{pmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 3-c_1 \\ 1-c_2 \\ -1-c_3 \end{pmatrix}$$

multipily & comapre

$b_2c_3 - b_3c_2 = c_1 - 3$ ....(1)

$c_3 - b_3c_1 = 1 - c_2$ ....(2)

$c_2 - b_2c_1 = 1 + c_3$ ....(3)

$(1)\hat{i} - (2)\hat{j} + (3)\hat{k}$

$\hat{i}(b_2c_3 - b_3c_2) - \hat{j}(c_3 - b_3c_1) + \hat{k}(c_2 - b_2c_1)$

$= c_1\hat{i} + c_2\hat{j} + c_2\hat{k} - 3\hat{i} - \hat{j} + \hat{k}$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Take dot product with $\vec{b}$

$0 = \vec{c}.\vec{b} - \vec{a}.\vec{b}$

$\vec{b}.\vec{c} = 0$

$\vec{b} \perp \vec{c}$

$\vec{b} \wedge \vec{c} = 90^o$

Take dot product with $\vec{c}$

$0 = |\vec{c}|^2 - \vec{a}.\vec{c}$

$\vec{a}.\vec{c} = |\vec{c}|^2$

$\vec{a}.\vec{c} \neq 0$

$\vec{b} \times \vec{c} = \vec{c} - \vec{a}$

Squaring

$|\vec{b}|^2 \, |\vec{c}|^2 = |\vec{c}|^2 + |\vec{a}|^2 - 2\vec{c}.\vec{a}$

$|\vec{b}|^2 \, |\vec{c}|^2 = |\vec{c}|^2 + 11 - 2|\vec{c}|^2$

$|\vec{b}|^2 \, |\vec{c}|^2 = 11 - |\vec{c}|^2$

$|\vec{c}|^2 = \frac{11}{|\vec{b}|^2 + 1}$

$|\vec{c}| \le \sqrt{11}$

given $\vec{a}.\vec{b} = 0$

$b_2 - b_3 = -3$ also

$b_2^2 + b_3^2 - 2b_2b_3 = 9 \quad b_2b_3 > 0$

$b_2^2 + b_3^2 = 9 + 2b_2b_3$

$b_2^2 + b_3^2 = 9 + 2b_2b_3 > 9$

$b_2^2 + b_3^2 > 9$

$|\vec{b}| = \sqrt{1 + b_2^2 + b_3^2}$

$|\vec{b}| > \sqrt{10}$

14. For $x \in R$, let the function y(x) be the solution of the differential equation $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi}{12}x\right), y(0) = 0$.

Then, which of the following statements is/are TRUE ?

(A) y(x) is an increasing function

(B) y(x) is decreasing function

(C) There exists a real number β such that the line y = β intersects the curve y = y(x) at infinitely many points

(D) y(x) is a periodic function

Ans. (C)

Sol. $\frac{dy}{dx} + 12y = \cos\left(\frac{\pi x}{12}\right)$

I.F. $= e^{12x} \Rightarrow y.e^{12x} = \int e^{12x} \cdot \cos\left(\frac{\pi x}{12}\right) dx + C$

$\Rightarrow y \cdot e^{12x} = \frac{e^{12x}}{12^2 + \left(\frac{\pi}{12}\right)^2}\left[12\cos\frac{\pi x}{12} + \frac{\pi}{12}\sin\left(\frac{\pi x}{12}\right)\right]$

$\because y(0) = 0 \Rightarrow C = -\frac{12}{12^2 + \left(\frac{\pi}{12}\right)^2}$

So $y = \frac{1}{\lambda}[\underbrace{12\cos\left(\frac{\pi x}{12}\right) + \frac{\pi}{12}\sin\left(\frac{\pi x}{12}\right)}_{f_1(x)} - 12e^{-12x}]$

$$\frac{dy}{dx}=\frac{1}{\lambda}[\underbrace{-\pi\sin\left(\frac{\pi x}{12}\right)+\frac{\pi^2}{12^2}\cos\left(\frac{\pi x}{12}\right)}_{f_2(x)}+12e^{-12x}]$$

When x is large then $12e^{-12x}$ tends to zero.

But $f_2(x)$ varies in $\left[-\sqrt{\pi^2+\left(\frac{\pi}{12}\right)^4},\sqrt{\pi^2+\left(\frac{\pi}{12}\right)^4}\right]$

Hence $\frac{dy}{dx}$ is changing its sign.

So y(x) is non monotonic for all real number.
Also when x is very large then again $-12e^{-12x}$ is almost zero but $f_1(x)$ is periodic, so there exist some β for which $y=\beta$ intersect $y=y(x)$ at infinitely many points.

SECTION 3 (Maximum marks: 12)

- This section contains FOUR (04) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONLY ONE of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 If ONLY the correct option is chosen;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : -1 In all other cases.

15. Consider 4 boxes, where each box contains 3 red balls and 2 blue balls. Assume that all 20 balls are distinct. In how many different ways can 10 balls be chosen from these 4 boxes so that from each box at least one red ball and one blue ball are chosen ?

(A) 21816 (B) 85536 (C) 12096 (D) 156816

Ans. (A)

Sol.

3B	3B	3B	3B
2R	2R	2R	2R

$$\begin{bmatrix} 2B & 2B & 1B & 1B \\ 1R & 1R & 1R & 1R \end{bmatrix}\times 6$$

$$\begin{bmatrix} 3B & 1B & 1B & 1B \\ 1R & 1R & 1R & 1R \end{bmatrix}\times 4$$

$$\begin{bmatrix} 1B & 1B & 1B & 1B \\ 2R & 2R & 1R & 1R \end{bmatrix}\times 6$$

$$\begin{bmatrix} 2B & 1B & 1B & 1B \\ 2R & 1R & 1R & 1R \end{bmatrix}\times 6$$

Now

$$\left({}^3C_2\times {}^2C_1 \times {}^3C_2\times {}^2C_1\times {}^3C_1 \times {}^2C_1\times {}^3C_1\times {}^2C_1\right)\times 6+$$

$$\left({}^3C_3\times {}^2C_1 \times {}^3C_1\times {}^2C_1\times {}^3C_1 \times {}^2C_1\times {}^3C_1\times {}^2C_1\right)\times 4$$

$$+\left({}^3C_1\times {}^2C_2 \times {}^3C_1\times {}^2C_2\times {}^3C_1 \times {}^2C_1\times {}^3C_1\times {}^2C_1\right)\times 6+$$

$$\left({}^3C_2\times {}^2C_2 \times {}^3C_1\times {}^2C_1\times {}^3C_1 \times {}^2C_1\times {}^3C_1\times {}^2C_1\right)\times 16$$

$= 6^3\,[36 + 8 + 9 + 48] = 216 \times (53 + 48) = 216 \times 101 = 21816$

Alternative :

We needed to find coeff of

$x^4 y^6$, $x^5 y^5$ & $x^6 y^4$ in

$$\left({}^3C_1x+ {}^3C_2x^2+ {}^3C_3x^3\right)^4\left({}^2C_1y+ {}^2C_2y^2\right)^4$$

which is equivalent 0 coeff. of

y^2, xy, x^2 in

$\left(3+3x+x^2\right)^4\left(2+y\right)^4$

$= {}^4C_2 \times 2^2 \times 3^4$

$+ {}^4C_2 \times 3^4 \times {}^4C_1 \times 2^3$

$+$

${}^4C_2 \times 3^4 \times 2^4$

$+$

${}^4C_1 \times 3^3 \times 2^4 = 3^4\left(2^4 + 128 + 96 + \frac{64}{3}\right)$

$= 21816$

16. If $M = \begin{pmatrix} \frac{5}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, then which of the following matrices is equal to M^{2022} ?

(A) $\begin{pmatrix} 3034 & 3033 \\ -3033 & -3032 \end{pmatrix}$

(B) $\begin{pmatrix} 3034 & -3033 \\ 3033 & -3032 \end{pmatrix}$

(C) $\begin{pmatrix} 3033 & 3032 \\ -3032 & -3031 \end{pmatrix}$

(D) $\begin{pmatrix} 3032 & 3031 \\ -3031 & -3030 \end{pmatrix}$

Ans. (A)

Sol. Let M = B + I

$B = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}$, $B^2 = 0$ and hence $B^3, B^4 ..... = 0$

$M^{2022} = (B + I)^{2022} = I + 2022\ B = \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$

17. Suppose that

Box-I contains 8 red, 3 blue and 5 green balls,

Box-II contains 24 red, 9 blue and 15 green balls,

Box-III contains 1 blue, 12 green and 3 yellow balls,

Box-IV contains 10 green, 16 orange and 6 white balls.

A ball is chosen randomly from Box-I; call this ball b. If b is red then a ball is chosen randomly from Box-II, if b is blue then a ball is chosen randomly from Box-III, and if b is green then a ball is chosen randomly from Box-IV. The conditional probability of the event 'one of the chosen balls is white' given that the event 'at least one of the chosen balls is green' has happened, is equal to

(A) $\frac{15}{256}$ (B) $\frac{3}{16}$ (C) $\frac{5}{52}$ (D) $\frac{1}{8}$

Ans. (C)

Sol. Box I 8(R) 3(B) 5(G)

Box II 24(R) 9(B) 15(G)

Box III 1(R) 12(G) 3(y)

Box IV 10(G) 16(o) 6(w)

A (one of the chosen balls is white)

B (at least one of the chosen ball is given)

$$P\left(\frac{A}{N}\right)=\frac{P(A\cap B)}{P(B)}$$

$$A\cap B\rightarrow(wG)$$

$$B\rightarrow(GG,GR,GB,Gw)$$

$$=\frac{\frac{5}{16}\times\frac{6}{32}}{\frac{5}{16}\times1+\frac{8}{16}\times\frac{15}{48}+\frac{3}{16}\times\frac{12}{16}}$$

$$=\frac{15}{16}=\frac{5}{52}$$

18. For positive integer n, define $f(n)=n+\frac{16+5n-3n^2}{4n+3n^2}+\frac{32+n-3n^2}{8n+3n^2}+\frac{48-3n-3n^2}{12n+3n^2}+....+\frac{25n-7n^2}{7n^2}$.

Then, the value of $\lim_{n\to\infty} f(n)$ is equal to :

(A) $3+\frac{4}{3}\log_e 7$ (B) $4-\frac{3}{4}\log_e\left(\frac{7}{3}\right)$ (C) $4-\frac{4}{3}\log_e\left(\frac{7}{3}\right)$ (D) $3+\frac{3}{4}\log_e 7$

Ans. (B)

Sol. $f(n)=n+\sum_{r=1}^{n}\frac{16r+(9-4r)n-3n^2}{4rn+3n^2}$

$=(n)+\sum_{r=1}^{n}\frac{16r+9n-3n^2-4rn}{4rn+3n^2}$

$=(n)+\sum_{r=1}^{n}\left(\frac{16r+9n}{4rn+3n^2}-1\right)$

$=(n)+\sum_{r=1}^{n}\frac{16r+9n}{4rn+3n^2}-n$

Now $\underset{n\to\infty}{\ell t} f(n)=\underset{n\to\infty}{\ell t}\sum_{r=1}^{n}\frac{16r+9n}{4rn+3n^2}=\underset{n\to\infty}{\ell t}\sum_{r=1}^{n}\frac{\left(16\frac{r}{n}+9\right)\frac{1}{n}}{4\frac{r}{n}+3}$

$=\int_0^1\frac{16x+9}{4x+3}dx$

$=\int_0^1\left[\frac{4(4x+3)}{4x+3}-\frac{3}{4x+3}\right]dx$

$=\int_0^1 4dx-\int_0^1\frac{3}{4x+3}dx$

$=4(x)_0^1-\frac{3}{4}(\ell n\,|4x+3|)_0^1=4-\frac{3}{4}\ell n\left(\frac{7}{3}\right)$