JEE Adv. 2022

Question Paper With Text Solution

28 August | Paper-1

MATHS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE ADV. AUGUST 2022 | 28TH. AUGUST PAPER-1

SECTION 1 (Maximum Marks: 24)

- This section contains EIGHT (08) questions.
- The answer to each question is a NUMERICAL VALUE.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

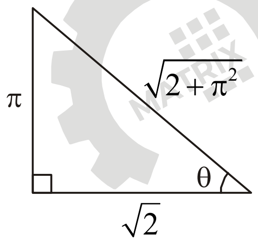
  Full Marks : +3 ONLY if the correct numerical value is entered;

  Zero Marks : 0 In all other cases.

1. Considering only the principal values of the inverse trigonometric functions, the value of $\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}+\frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2}+\tan^{-1}\frac{\sqrt{2}}{\pi}$ is ____.

Ans. (2.35 or 2.36)

Sol. $\cos^{-1}\sqrt{\frac{2}{2+\pi^2}}=\tan^{-1}\frac{\pi}{\sqrt{2}}$



$$\sin^{-1}\left(\frac{2\sqrt{2}\pi}{2+\pi^2}\right)=\sin^{-1}\left(\frac{2\times\frac{\pi}{\sqrt{2}}}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}\right)$$

$$=\pi-2\tan^{-1}\left(\frac{\pi}{\sqrt{2}}\right)$$

$$\left(\text{As, }\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\pi-2\tan^{-1}x, x\geq 1\right)$$

and $\tan^{-1}\dfrac{\sqrt{2}}{\pi}=\cot^{-1}\left(\dfrac{\pi}{\sqrt{2}}\right)$

$\therefore$ Expression $=\dfrac{3}{2}\left(\tan^{-1}\dfrac{\pi}{\sqrt{2}}\right)+\dfrac{1}{4}\left(\pi-2\tan^{-1}\dfrac{\pi}{\sqrt{2}}\right)+\cot^{-1}\left(\dfrac{\pi}{\sqrt{2}}\right)$

$=\left(\dfrac{3}{2}-\dfrac{2}{4}\right)\tan^{-1}\dfrac{\pi}{\sqrt{2}}+\dfrac{\pi}{4}+\cot^{-1}\dfrac{\pi}{\sqrt{2}}$

$=\left(\tan^{-1}\dfrac{\pi}{\sqrt{2}}+\cot^{-1}\dfrac{\pi}{\sqrt{2}}\right)+\dfrac{\pi}{4}$

$=\dfrac{\pi}{2}+\dfrac{\pi}{4}=\dfrac{3\pi}{4}$

= 2.35 or 2.36

2. Let α be a positive real number. Let $f:R\to R$ and g : $(\alpha, \infty)\to R$ be the functions defined by $f(x)=\sin\left(\dfrac{\pi x}{12}\right)$ and $g(x)=\dfrac{2\log_e\left(\sqrt{x}-\sqrt{\alpha}\right)}{\log_e\left(e^{\sqrt{x}}-e^{\sqrt{\alpha}}\right)}$. Then the value of $\lim\limits_{x\to\alpha^+} f(g(x))$ is ______.

Ans. (00.50)

Sol. $\lim\limits_{x\to\alpha^+} g(x)=\lim\limits_{x\to\alpha^+}\dfrac{\dfrac{2}{\sqrt{x}-\sqrt{\alpha}}\left(\dfrac{1}{2\sqrt{x}}\right)}{\dfrac{1}{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}\left(\dfrac{1}{2\sqrt{x}}\cdot e^{\sqrt{x}}\right)}$ **(Using L'Hospital Rule)**

$=\lim\limits_{x\to\alpha^+}\dfrac{e^{\sqrt{x}}-e^{\sqrt{\alpha}}}{\sqrt{x}-\sqrt{\alpha}}\cdot\dfrac{1}{e^{\sqrt{x}}}\cdot 2$

$=\lim\limits_{x\to\alpha^+}\dfrac{e^{\sqrt{x}}\cdot\dfrac{1}{2\sqrt{x}}}{\dfrac{1}{2\sqrt{x}}}\cdot\dfrac{2}{e^{\sqrt{x}}}=2$

$\lim\limits_{x\to\alpha^+} f(g(x))=f\left(\lim\limits_{x\to\alpha^+} g(x)\right)=\sin\dfrac{\pi}{6}=\dfrac{1}{2}=00.50$

3. In a study about a pandemic, data of 900 persons was collected. It was found that

190 persons had symptom of fever,

220 persons had symptom of cough,

220 persons had symptom of breathing problem,

330 persons had symptom of fever or cough or both,

350 persons had symptom of cough or breathing problem or both,

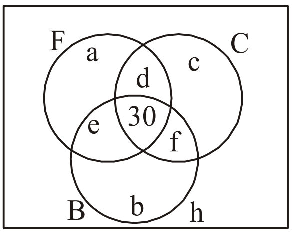
340 persons had symptom of fever or breathing problem or both,

30 persons had all three symptoms (fever, cough and breathing problem).

If a person is chosen randomly from these 900 persons, then the probability that the person has at most one symptom is _____________

Ans. (00.80)

Sol.



a + e + d = 160
c + d + f = 190
b + e + f = 190
a + b + c + 2(d + e + f) = 540

$$\begin{cases} a+d+c+e+f=300 \\ b+c+d+f+e=320 \\ a+b+d+e+f=310 \end{cases}$$

x+ 2y = 540
2x + 3y = 930
2x + 4y = 1080
y = 150
x = 240

$$2(\underset{x}{a+b+c})+3(\underset{y}{e+f+g})=930$$

$$\underset{240}{a+b+c}+\underset{150+30+h=900}{d+e+f}=900$$

h = 480

$$=\frac{a+b+c+h}{900}=\frac{240+480}{900}=\frac{720}{900}=0.80$$

4. Let z be a complex number with non-zero imaginary part. If $\dfrac{2+3z+4z^2}{2-3z+4z^2}$ is a real number, then the value of $|z|^2$ is _____________

Ans. (0.50)

Sol. Let $w=\dfrac{4z^2+3z+2}{4z^2-3z+2}=1+\dfrac{6z}{4z^2-3z+2}$

$\Rightarrow \quad w=1+\dfrac{6}{2\left(2z+\dfrac{1}{z}\right)-3}$

$\because$ $w\in R$ then $2z+\dfrac{1}{z}\in R$

$\Rightarrow \quad 2z+\dfrac{1}{z}=2\bar{z}+\dfrac{1}{\bar{z}}$

$\Rightarrow \quad 2(z-\bar{z})-\dfrac{z-\bar{z}}{|z|^2}=0$

$\Rightarrow \quad (z-\bar{z})\left(2-\dfrac{1}{|z|^2}\right)=0$

$\because \quad z\neq\bar{z}$ (given)

So $|z|^2=\dfrac{1}{2}$

5. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $i=\sqrt{-1}$. In the set of complex numbers, the number of distinct roots of the equation $\bar{z}-z^2=i\left(\bar{z}+z^2\right)$ is_______

Ans. (4.00)

Sol. $\bar{z}-z^2=i\left(\bar{z}+z^2\right)$

$\Rightarrow (1-i)\bar{z}=(1+i)z^2$

$\Rightarrow \dfrac{(1-i)}{(1+i)}\bar{z}=z^2$

$\Rightarrow \left(-\dfrac{2i}{2}\right)\bar{z}=z^2$

$\therefore z^2=-i\bar{z}$

Let $z=x+iy$,

$\therefore$ $(x^2-y^2)+i(2xy)=-i(x-iy)$

so, $x^2-y^2+y=0$ _______(1)

and $(2y+1)x=0$ _______(2)

$\therefore$ $x=0$ or $y=-\dfrac{1}{2}$

Case I : When $x=0$

$\therefore$ (1) $\Rightarrow y(1-y)=0 \therefore y=0,1$
$\therefore$ (0,0), (0,1)

Case II : When $y=-\frac{1}{2}$

$\therefore (1) \Rightarrow x^2-\frac{1}{4}-\frac{1}{2}=0 \Rightarrow x^2=\frac{3}{4} \Rightarrow x=\pm\frac{\sqrt{3}}{2}$

$\therefore \left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right),\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

$\Rightarrow$ Number of distinct 'z' is equal to 4.

6. Let $l_1, l_2, \ldots, l_{100}$ be consecutive terms of an arithmetic progression with common difference d_1, and let $w_1, w_2, \ldots, w_{100}$ be consecutive terms of another arithmetic progression with common difference d_2, where $d_1 d_2 = 10$. For each $i = 1, 2, \ldots, 100$, let R_i be a rectangle with length l_i, width w_i and area A_i. If $A_{51} - A_{50} = 1000$, then the value of $A_{100} - A_{90}$ is ____________.

Ans. (18900)

Sol. $A_{51}-A_{50}=1000$

$\ell_{51} w_{51}-\ell_{50} w_{50}=1000$

$(\ell_1+50d_1)(w_1+50d_2)-(\ell_1+49d_1)(w_1+49d_2)=1000$

$(50-49)\ell_1 d_2+(50-49)w_1 d_1+(50^2-49^2)d_1 d_2=1000$

$\ell_1 d_2+w_1 d_1+99d_1 d_2=1000$

$\ell_1 d_2+w_1 d_1=10$

Now $A_{100}-A_{90}$

$=\ell_{100} w_{100}-\ell_{90} w_{90}$

$=(\ell_1+99d_1)(w_1+99d_2)-(\ell_1+89d_1)(w_1+89d_2)$

$=(99-89)\ell_1 d_2+(99-89)W_1 d_1+(99^2-89^2)d_1 d_2$

$=10\ell_1 d_2+10w_1 d_1+(10)(188)d_1 d_2$

$= 10 \times 10 + 100 \times 188$

$= 100\ (189)$

$= 18900$

7. The number of 4-digit integers in the closed interval [2022, 4482] formed by using the digits 0, 2, 3, 4, 6, 7 is ____________.

Ans. (569)

Sol. (1)

2	0	2	2, 3 4,6,7

$\longrightarrow 5$

(2)

2	0	3, 4 6, 7	
		↓ 4	↓ 6

$\longrightarrow 24$

(3)

2	2, 3, 4 6, 7		
	↓ 5	↓ 6	↓ 6

$\longrightarrow 180$

(4)

3			
	↓ 6	↓ 6	↓ 6

$\longrightarrow 216$

(5)

4	0, 2, 3, 4		
	↓ 4	↓ 6	↓ 6

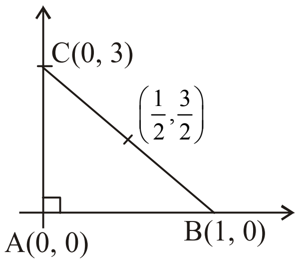
$\longrightarrow 144$

Number of 4 digit integers in [2022,4482]
$= 5 + 24 + 180 + 216 + 144 = 569$

8. Let ABC be the triangle with AB = 1, AC = 3 and $\angle BAC = \frac{\pi}{2}$. If a circle of radius $r > 0$ touches the sides AB, AC and also touches internally the circumcircle of the triangle ABC, then the value of r is _____________.

Ans. (0.83 or 0.84)

Sol. Let A be the origin B on x-axis, C on y-axis as shown below



$\therefore$ Equation of circumcircle is

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\left(\frac{1}{2}\right)^2+\left(\frac{3}{2}\right)^2=\frac{5}{2} \qquad \_\_\_\_\_\_(1)$$

Required circle touches AB and AC, have radius r

$\therefore$ Equation be $(x-r)^2+(y-r)^2=r^2$ $\qquad$ ________(2)

If circle in equation (2) touches circumcircle internally, we have

$d_{c_1c_2} = |r_1 - r_2|$

$$\Rightarrow \left(\frac{1}{2}-r\right)^2 + \left(\frac{3}{2}-r\right)^2 = \left(\left|\sqrt{\frac{5}{2}}-r\right|\right)^2$$

$$\Rightarrow \frac{1}{4}+r^2-r+\frac{9}{4}+r^2-3r = \left(\sqrt{\frac{5}{2}}-r\right)^2 \text{ or } \left(r-\sqrt{\frac{5}{2}}\right)^2$$

$$\Rightarrow 2r^2-4r+\frac{5}{2} = \frac{5}{2}+r^2-\sqrt{10}r$$

$\Rightarrow r = 0$ or $4-\sqrt{10}$

$\Rightarrow r = 0.837$

SECTION 2 (Maximum Marks: 24)

- This section contains SIX (06) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is(are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 ONLY if (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct; Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –2 In all other cases.

9. Consider the equation

$$\int_1^e \frac{(\log_e x)^{1/2}}{x\left(a-(\log_e x)^{3/2}\right)^2}dx = 1, a \in (-\infty, 0) \cup (1, \infty)$$

Which of the following statements is/are TRUE ?

(A) No a satisfies the above equation

(B) An integer a satisfies the above equation

(C) An irrational number a satisfies the above equation

(D) More than one a safisfy the above equation

Ans. (C, D)

Sol. Let $I = \int_1^e \frac{(\ln x)^{1/2}dx}{x\left(a-(\ln x)^{3/2}\right)^2}$

put $a-(\ln x)^{3/2} = t$

$\Rightarrow \quad -\frac{3}{2}(\ln x)^{1/2}\cdot\frac{1}{x}dx = dt$

$\therefore \quad I = \int_a^{a-1}\frac{\left(-\frac{2}{3}\right)dt}{t^2}$

$= \left(-\frac{2}{3}\right)\frac{t^{-2+1}}{-2+1}\Big|_a^{a-1}$

$= \frac{2}{3t}\Big|_a^{a-1} = \frac{2}{3}\left(\frac{1}{a-1}-\frac{1}{a}\right)$

$\therefore \quad I = \left(\frac{2}{3}\right)\frac{1}{a(a-1)} = 1$

$\Rightarrow 2 = 3a^2 - 3a$

$\Rightarrow 3a^2 - 3a - 2 = 0$

$\Rightarrow \quad a = \frac{3 \pm \sqrt{9-4(3)(-2)}}{6}$

$a = \frac{3+\sqrt{33}}{6}, \frac{3-\sqrt{33}}{6}$

10. Let $a_1, a_2, a_3, \ldots$ be an arithmetic progression with $a_1 = 7$ and common difference 8.

Let $T_1, T_2, T_3, \ldots$ be such that $T_1 = 3$ and $T_{n+1} - T_n = a_n$ for $n \geq 1$. Then, which of the following is/are TRUE ?

(A) $T_{20} = 1604$ (B) $\sum_{k=1}^{20} T_k = 10510$ (C) $T_{30} = 3454$ (D) $\sum_{k=1}^{30} T_k = 35610$

Ans. (BC)

Sol. $a_1 = 7, d = 8$

$T_{n+1} - T_n = a_n \; \forall n \geq 1$

$S_n = T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$

$S_n = \quad T_1 + T_2 + T_3 + \ldots + T_{n-1} + T_n$

on subtraction

$T_n = T_1 + a_1 + a_2 + \ldots + a_{n-1}$

$T_n = 3 + (n-1)(4n-1)$

$T_n = 4n^2 - 5n + 4$

$\sum_{k=1}^{n} T_k = 4\sum n^2 - 5\sum n + 4n$

$T_{20} = 1504$

$T_{30} = 3454$

$\sum_{k=1}^{30} T_k = 35615$

$\sum_{k=1}^{20} T_k = 10510$

11. Let P_1 and P_2 be two planes given by

$$P_1 : 10x + 15y + 12z - 60 = 0,$$

$$P_2 : -2x + 5y + 4z - 20 = 0,$$

Which of the following straight lines can be an edge of some tetrahedron whose two faces lie on P_1 and P_2 ?

(A) $\frac{x-1}{0} = \frac{y-1}{0} = \frac{z-1}{5}$ (B) $\frac{x-6}{-5} = \frac{y}{2} = \frac{z}{3}$

(C) $\frac{x}{-2} = \frac{y-4}{5} = \frac{z}{4}$ (D) $\frac{x}{1} = \frac{y-4}{-2} = \frac{z}{3}$

Ans. (A, B, D)

Sol. Edge of Tetrahedron can be (i) common intersection line
(ii) Line totally in P_2/P_1 and meeting intersecting line.
(iii) Line intersecting both P_1 and P_2

(i) Common intersecting line is parallel to $\vec{n}_1 \times \vec{n}_2$

$$10a + 15b + 12c = 0$$
$$-2a+5b+4c = 0$$

i.e. it has direction ratio 0, –4, 5 and a point on it can be (0, 0, 5)

$\frac{x}{0} = \frac{y}{-4} = \frac{z-5}{5}$ Not possible

(ii) Line in P_2/P_1 must be perpendicular to their normals. option 'D' Line in plane P_2 and it should also cut common line

$\Rightarrow$ $0, -4\lambda, 5\lambda, +5$ should lie on D.

$\frac{0}{1} = \frac{-4\lambda-4}{-2} = \frac{5\lambda+5}{3}$ which is true for $\lambda = -1$.

Hence 'D' is correct.

(iii) In option 'C' is parallel to normal at 'P_2' but it is passing through (0, 4, 0) which is a point on intersecting lines. Hence, this can't be an edge of tetrahedron (A) and (B) cuts both P_1 and P_2 and do not passes through common intersecting lines hence are possible.

(A) and (B) are possible, Hence A, B, D are correct

12. Let S be the reflection of a point Q with respect to the plane given by $\vec{r} = -(t+p)\hat{i} + t\hat{j} + (1+p)\hat{k}$ where t, p are real parameters and $\hat{i}$, $\hat{j}$, $\hat{k}$ are the unit vectors along the three positive coordinate axes. If the position vectors of Q and S are $10\hat{i} + 15\hat{j} + 20\hat{k}$ and $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$ respectively, then which of the following is/are TRUE ?

(A) $3(\alpha+\beta) = -101$ (B) $3(\beta+\gamma) = -71$

(C) $3(\gamma+\alpha) = -86$ (D) $3(\alpha+\beta+\gamma) = -121$

Ans. (A, B, C)

Sol. $\vec{r} = \hat{k} + (-\hat{i}+\hat{k})p + t(-\hat{i}+\hat{j})$

$$\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

Equation of Plane $x + y + z = 1$

$Q = (10, 15, 20)$
$F = (10+d, 15+d, 20+d)$
$10+d + 10+d + 20+d = 1$
$3d = -44$

$$\frac{\alpha+10}{2} = 10 + d$$

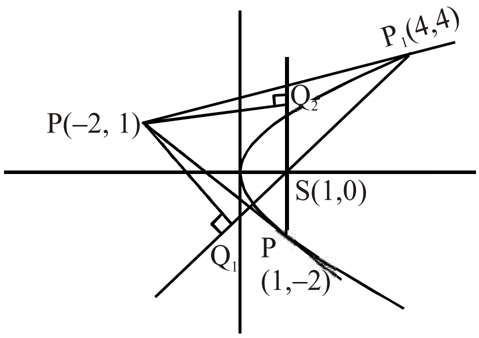
$3\alpha = 30 - 2(44) = -58$
$3\beta = 45 - 88 = -43$
$3\gamma = 60 - 88 = -28$

13. Consider the parabola $y^2 = 4x$. Let S be the focus of the parabola. A pair of tangents drawn to the parabola from the point $P = (-2, 1)$ meet the parabola at P_1 and P_2. Let Q_1 and Q_2 be points on the lines SP_1 and SP_2 respectively such that PQ_1 is perpendicular to SP_1 and PQ_2 is perpendicular to SP_2. Then, which of the following is/are TRUE ?

(A) $SQ_1 = 2$ (B) $Q_1Q_2 = \frac{3\sqrt{10}}{5}$ (C) $PQ_1 = 3$ (D) $SQ_2 = 1$

Ans. (B,C,D)

Sol. Let equation of tangent with slope 'm' be



$T : y = mx + \frac{1}{m}$

T : passes through (–2, 1) so

$1 = -2m + \frac{1}{m}$

$\Rightarrow m = -1$ or $m = \frac{1}{2}$

Points are given by $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

So, one point will be (1, –2) and (4, 4)

$P_1S : 4x - 3y - 4 = 0$

$P_2S : x - 1 = 0$

$PQ_1 = \left|\frac{4(-2) - 3(1) - 4}{5}\right| = 3$

$SP = \sqrt{10}; PQ_2 = 3; SQ_1 = 1 = SQ_2$

$\frac{1}{2}\left(\frac{Q_1Q_2}{2}\right) \times \sqrt{10} = \frac{1}{2} \times 3 \times 1$ (comparing Areas)

$\Rightarrow Q_1Q_2 = \frac{2 \times 3}{\sqrt{10}} = \frac{3\sqrt{10}}{5}$

14. Let |M| denote the determinant of a square matrix. M. Let $g : \left[0, \frac{\pi}{2}\right] \to R$ be the function defined by

$$g(\theta) = \sqrt{f(\theta) - 1} + \sqrt{f\left(\frac{\pi}{2} - \theta\right) - 1}$$

where $f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} + \begin{vmatrix} \sin\pi & \cos\left(\theta + \frac{\pi}{4}\right) & \tan\left(\theta - \frac{\pi}{4}\right) \\ \sin\left(\theta - \frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right) \\ \cot\left(\theta + \frac{\pi}{4}\right) & \log_e\left(\frac{\pi}{4}\right) & \tan\pi \end{vmatrix}$.

Let p(x) be a quadratic polynomial whose roots are the maximum and minimum values of the function g(θ), and $p(2) = 2 - \sqrt{2}$. Then, which of the following is/are TRUE ?

(A) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$ (B) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$ (C) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$ (D) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

Ans. (A, C)

Sol. $$f(\theta)=\frac{1}{2}\begin{vmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{vmatrix}+\begin{vmatrix}\sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\frac{\pi}{2} & \log_e\left(\frac{4}{\pi}\right)\\ \cot\left(\theta+\frac{\pi}{4}\right) & \log_e\frac{\pi}{4} & \tan\pi\end{vmatrix}$$

$$f(\theta)=\frac{1}{2}\begin{vmatrix}2 & \sin\theta & 1\\ 0 & 1 & \sin\theta\\ 0 & -\sin\theta & 1\end{vmatrix}+\begin{vmatrix}0 & -\sin\left(\theta-\frac{\pi}{4}\right) & \tan\left(\theta-\frac{\pi}{4}\right)\\ \sin\left(\theta-\frac{\pi}{4}\right) & 0 & \log_e\left(\frac{4}{\pi}\right)\\ -\tan\left(\theta-\frac{\pi}{4}\right) & -\log_e\left(\frac{4}{\pi}\right) & 0\end{vmatrix}$$

$f(\theta)=(1+\sin^2\theta)+0$ (skew symmetric)

$$g(\theta)=\sqrt{f(\theta)-1}+\sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$$

$=|\sin\theta|+|\cos\theta|$ for $\theta\in\left[0,\frac{\pi}{2}\right]$

$g(\theta)\in\left[1,\sqrt{2}\right]$

Again let $P(x)=k(x-\sqrt{2})(x-1)$

$2-\sqrt{2}=k(2-\sqrt{2})(2-1)$

$\Rightarrow k=1$ ($P(2)=2-\sqrt{2}$ given)

$\therefore P(x)=\left(x-\sqrt{2}\right)(x-1)$

for option (A) $P\left(\frac{3+\sqrt{2}}{4}\right)<0$ correct

option (B) $P\left(\frac{1+3\sqrt{2}}{4}\right)<0$ incorrect

option (C) $P\left(\frac{5\sqrt{2}-1}{4}\right)>0$ correct

option (D) $P\left(\frac{5-\sqrt{2}}{4}\right)>0$ incorrect

SECTION 3 (Maximum Marks: 12)

- This section contains FOUR (04) Matching List Sets.
- Each set has ONE Multiple Choice Question.
- Each set has TWO lists: List-I and List-II.
- List-I has Four entries (I), (II), (III) and (IV) and List-II has Five entries (P), (Q), (R), (S) and (T).
- FOUR options are given in each Multiple Choice Question based on List-I and List-II and ONLY ONE of these four options satisfies the condition asked in the Multiple Choice Question.
- Answer to each question will be evaluated according to the following marking scheme:
  Full Marks : +3 ONLY if the option corresponding to the correct combination is chosen;
  Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);
  Negative Marks : -1 In all other cases.

15. Consider the following lists :

List - I	**List - II**
(I) $\left\{x \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] : \cos x + \sin x = 1\right\}$	(P) has two elements
(II) $\left\{x \in \left[-\frac{5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$	(Q) has three elements
(III) $\left\{x \in \left[-\frac{6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos(2x) = \sqrt{3}\right\}$	(R) has four elements
(IV) $\left\{x \in \left[-\frac{7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$	(S) has five elements
	(T) has six elements

The correct option is:

(A) (I) $\rightarrow$ (P); (II) $\rightarrow$ (S); (III) $\rightarrow$ (P); (IV) $\rightarrow$ (S)

(B) (I) $\rightarrow$ (P); (II) $\rightarrow$ (P); (III) $\rightarrow$ (T); (IV) $\rightarrow$ (R)

(C) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (P); (III) $\rightarrow$ (T); (IV) $\rightarrow$ (S)

(D) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (S); (III) $\rightarrow$ (P); (IV) $\rightarrow$ (R)

Ans. (B)

Sol. $\left\{x \in \left[\frac{-2\pi}{3}, \frac{2\pi}{3}\right], \cos x + \sin x = 1\right\}$

$\cos x + \sin x = 1$

$\sin\left(\frac{\pi}{4} + x\right) = \frac{1}{\sqrt{2}}$

$\frac{\pi}{4} + x = n\pi + (-1)^n \frac{\pi}{4}$

$x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$

$\therefore$ x has 2 elements. $\rightarrow$ P

(ii) $\left\{x \in \left[\frac{-5\pi}{18}, \frac{5\pi}{18}\right] : \sqrt{3}\tan 3x = 1\right\}$

$\sqrt{3}\tan 3x = 1$

$\tan 3x = \frac{1}{\sqrt{3}}$

$3x = n\pi + \frac{\pi}{6}$

$x = \frac{n\pi}{3} + \frac{\pi}{18}$

$\therefore$ x has 2 elements. $\rightarrow$ P

(iii) $\left\{x \in \left[\frac{-6\pi}{5}, \frac{6\pi}{5}\right] : 2\cos 2x = \sqrt{3}\right\}$

$2\cos 2x = \sqrt{3}$

$\cos 2x = \frac{\sqrt{3}}{2}$

$2x = 2n\pi \pm \frac{\pi}{6}$

$x = n\pi \pm \frac{\pi}{12}$

$\therefore$ x has 6 elements. $\rightarrow$ T

(iv) $\left\{x \in \left[\frac{-7\pi}{4}, \frac{7\pi}{4}\right] : \sin x - \cos x = 1\right\}$

$\sin x - \cos x = 1$

$\sin\left(x-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

$x-\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}$

$x=n\pi+(-1)^n\frac{\pi}{4}+\frac{\pi}{4}$

$\therefore$ x has 4 elements. $\rightarrow$ R

$\therefore$ option B is correct.

16. Two players, P_1 and P_2, play a game against each other. In every round of the game, each player rolls a fair die once, where the six faces of the die have six distinct numbers. Let x and y denote the readings on the die rolled by P_1 and P_2, respectively. If $x > y$, then P_1 scores 5 points and P_2 scores 0 point. If $x = y$, then each player scores 2 points. If $x < y$, then P_1 scores 0 point and P_2 scores 5 points. Let X_i and Y_i be the total scores of P_1 and P_2, respectively, after playing the i^{th} round.

List - I	**List - II**
(I) Probability of $(X_2 \geq Y_2)$ is	(P) $\frac{3}{8}$
(II) Probability of $(X_2 > Y_2)$ is	(Q) $\frac{11}{16}$
(III) Probability of $(X_3 = Y_3)$ is	(R) $\frac{5}{16}$
(IV) Probability of $(X_3 > Y_3)$ is	(S) $\frac{355}{864}$
	(T) $\frac{77}{432}$

The correct option is:

(A) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (R); (III) $\rightarrow$ (T); (IV) $\rightarrow$ (S)

(B) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (R); (III) $\rightarrow$ (T); (IV) $\rightarrow$ (T)

(C) (I) $\rightarrow$ (P); (II) $\rightarrow$ (R); (III) $\rightarrow$ (Q); (IV) $\rightarrow$ (S)

(D) (I) $\rightarrow$ (P); (II) $\rightarrow$ (R); (III) $\rightarrow$ (Q); (IV) $\rightarrow$ (T)

Ans. (A)

Sol.

	P_1	P_2
P_1 win	5	0
Draw	2	2
P_1 Lose	0	5

Probability that P_1 wins $= \frac{15}{36} = \frac{5}{12}$

Probability that P_1 Lose $= \frac{5}{12}$

Probability that P_1 draw $= \frac{1}{6}$

Now $x_2 \geq y_2 \Rightarrow P_1$ (i) $WW \rightarrow \left(\frac{5}{12}\right)^2$

(ii) $WD \rightarrow \frac{5}{12} \times \frac{1}{6} \times 2$

(iii) $WL \rightarrow \frac{5}{12} \times \frac{5}{12} \times 2$

(iv) $DD \rightarrow \frac{1}{6} \times \frac{1}{6}$

Adding all $\frac{25+20+50+4}{144} = \frac{99}{144} = \frac{11}{16}$

$I \rightarrow Q$

Prob. $x_3 > y_3$

$\Rightarrow$ (i) $WWW \rightarrow \left(\frac{5}{12}\right)^3$

(ii) $WWL \rightarrow \left(\frac{5}{12}\right)^3 \times 3$

(iii) (i) $WWD \rightarrow \left(\frac{5}{12}\right)^2 \times \frac{1}{6} \times 3$

(iv) $WDD \rightarrow \frac{5}{12} \times \left(\frac{1}{6}\right)^2 \times 3$

Adding all $= \frac{125+375+150+60}{12^3} = \frac{710}{12 \times 144} = \frac{355}{864}$

$IV \rightarrow S$

Thus (A) is correct

17. Let p, q, r be nonzero real numbers that are, respectively, the 10^{th}, 100^{th} and 1000^{th} terms of a harmonic progression. Consider the system of linear equations

$$x + y + z = 1$$
$$10x + 100y + 1000z = 0$$
$$qr\,x + pr\,y + pq\,z = 0$$

List - I	**List - II**
(I) If $\frac{q}{r} = 10$, then the system of linear equations has	(P) $x = 0, y = \frac{10}{9}, z = -\frac{1}{9}$ as a solution
(II) If $\frac{p}{q} \neq 100$, then the system of linear equations has	(Q) $x = \frac{10}{9}, y = -\frac{1}{9}, z = 0$ as a solution
(III) If $\frac{p}{q} \neq 10$, then the system of linear equations has	(R) infinitely many solutions
(IV) If $\frac{p}{q} = 10$, then the system of linear equations has	(S) no solution
	(T) at least one solution

The correct option is:

(A) (I) $\rightarrow$ (T); (II) $\rightarrow$ (R); (III) $\rightarrow$ (S); (IV) $\rightarrow$ (T)

(B) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (S); (III) $\rightarrow$ (S); (IV) $\rightarrow$ (R)

(C) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (R); (III) $\rightarrow$ (P); (IV) $\rightarrow$ (R)

(D) (I) $\rightarrow$ (T); (II) $\rightarrow$ (S); (III) $\rightarrow$ (P); (IV) $\rightarrow$ (T)

Ans. (B)

Sol. $x + y + z = 1$ _______(1)

$10x + 100y + 1000z = 0$ _______(2)

$qrx + pry + pqz = 0$ _______(3)

Equation (3) can be re-written as

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 0 \qquad (\because\ p, q, r \neq 0)$$

Let $p = \frac{1}{a + 9d}, q = \frac{1}{a + 99d}, r = \frac{1}{a + 999d}$

Now, equation (3) is
(a + 9d)x + (a + 99d)y + (a + 999d)z = 0

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 100 & 1000 \\ a+9d & a+99d & a+999d \end{vmatrix} = 0$$

$$\Delta_x = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 100 & 1000 \\ 0 & a+99d & a+999d \end{vmatrix} = 900(d-a)$$

$$\Delta_y = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 0 & 1000 \\ a+9d & 0 & a+999d \end{vmatrix} = 990(a-d)$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 1 \\ 10 & 100 & 0 \\ a+9d & a+99d & 0 \end{vmatrix} = 90(d-a)$$

Option I: If $\frac{q}{r} = 10 \Rightarrow a = d$

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

And eq. (1) and eq. (2) represents non-parallel planes and eq. (2) and eq. (3) represents same plane
$\Rightarrow$ Infinitely many solutions
I $\rightarrow$ P, Q, R, T

Option II : $\frac{p}{r} \neq 100 \Rightarrow a \neq d$

$\Delta = 0, \Delta_x, \Delta_y, \Delta_z \neq 0$

No solution
II $\rightarrow$ S

Option III : $\frac{p}{q} \neq 10 \Rightarrow a \neq d$

No solution
III $\rightarrow$ S

Option IV : If $\frac{p}{q} = 10 \Rightarrow a = d$

Infinitely many solution
IV $\rightarrow$ P, Q, R, T

18. Consider the ellipse

$$\frac{x^2}{4}+\frac{y^2}{3}=1.$$

Let H(α, 0), $0 < \alpha < 2$, be a point. A straight line drawn through H parallel to the y-axis crosses the ellipse and its auxiliary circle at points E and F respectively, in the first quadrant. The tangent to the ellipse at the point E intersects the positive x-axis at a point G. Suppose the straight line joining F and the origin makes an angle ϕ with the positive x-axis.

List -I	**List -II**
(I) If $\phi=\frac{\pi}{4}$, then the area of the triangle FGH is	(P) $\frac{\left(\sqrt{3}-1\right)^4}{8}$
(II) If $\phi=\frac{\pi}{3}$, than the area of the triangle FGH is	(Q) 1
(III) If $\phi=\frac{\pi}{6}$, than the area of the triangle FGH is	(R) $\frac{3}{4}$
(IV) If $\phi=\frac{\pi}{12}$, than the area of the triangle FGH is	(S) $\frac{1}{2\sqrt{3}}$
	(T) $\frac{3\sqrt{3}}{2}$

The correct option is:

(A) (I) $\rightarrow$ (R); (II) $\rightarrow$ (S); (III) $\rightarrow$ (Q); (IV) $\rightarrow$ (P)

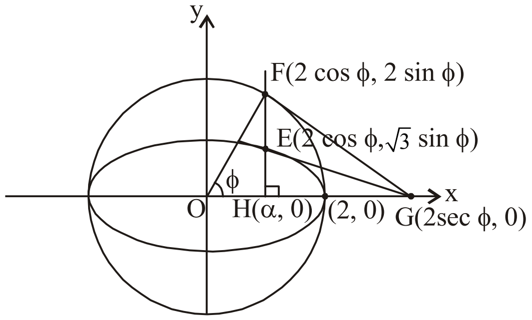
(B) (I) $\rightarrow$ (R); (II) $\rightarrow$ (T); (III) $\rightarrow$ (S); (IV) $\rightarrow$ (P)

(C) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (T); (III) $\rightarrow$ (S); (IV) $\rightarrow$ (P)

(D) (I) $\rightarrow$ (Q); (II) $\rightarrow$ (S); (III) $\rightarrow$ (Q); (IV) $\rightarrow$ (P

Ans. (C)

Sol.



$\alpha \equiv 2\cos\phi$

Tangent at E $(2\cos\phi, \sqrt{3}\sin\phi)$ to ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$

i.e. $\frac{x\cos\phi}{2}+\frac{y\sin\phi}{\sqrt{3}}=1$ intersect x-axis at G(2sec ϕ, 0)

Area of triangle $FGH=\frac{1}{2}(2\sec\phi-2\cos\phi)2\sin\phi$

$\Delta = 2\sin^2\phi.\tan\phi$

$\Delta = (1-\cos 2\phi)\cdot\tan\phi$

I. If $\phi=\frac{\pi}{4}, \Delta = 1 \rightarrow (Q)$

II. $\phi=\frac{\pi}{3}, \Delta = 2\cdot\left(\frac{\sqrt{3}}{2}\right)^2\cdot\sqrt{3}=\frac{3\sqrt{3}}{2}\rightarrow (T)$

III. If $\phi=\frac{\pi}{6}, \Delta = 2\cdot\left(\frac{1}{2}\right)^2\cdot\frac{1}{\sqrt{3}}=\frac{1}{2\sqrt{3}}\rightarrow (S)$

IV. If $\phi=\frac{\pi}{12}, \Delta = \left(1-\frac{\sqrt{3}}{2}\right)\cdot(2-\sqrt{3})=\frac{(2-\sqrt{3})^2}{2}\rightarrow (P)$