JEE Adv. October 2021

Question Paper With Text Solution

03 October. | Paper-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911
Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE ADV. OCTOBER 2021 | 03TR. OCTOBER PAPER-2

SECTION – A

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

  Zero Marks : 0 If unanswered;

  Negative Marks : -2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then

  choosing ONLY (A), (B) and (D) will get +4 marks;

  choosing ONLY (A) and (B) will get +2 marks;

  choosing ONLY (A) and (D) will get +2marks;

  choosing ONLY (B) and (D) will get +2 marks;

  choosing ONLY (A) will get +1 mark;

  choosing ONLY (B) will get +1 mark;

  choosing ONLY (D) will get +1 mark;

  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get -2 marks.

1. Let $S_1 = \{(i, j, k) : i, j, k \in \{1, 2, ...10\}\}$,

$S_2 = \{(i, j) : 1 \le i < j + 2 \le 10, i, j \in \{1, 2, ..., 10\}\}$

$S_3 = \{(i, j, k, \ell)\} : 1 \le i < j < k < \ell, i, j, k, \ell \in \{1, 2 ...., 10\}\}$

and $S_4 = \{(i, j, k, \ell)$: i, j, k and ℓ are distinct elements in $\{1, 2, ..., 10\}\}$.

If the total number of elements in the set S_r is n_r, r = 1, 2, 3, 4, then which of the following statements is (are) **TRUE ?**

(A) $n_1 = 1000$ (B) $n_2 = 44$ (C) $n_3 = 220$ (D) $\frac{n_4}{12} = 420$

Ans. (A,B,D)

Sol. $n_1 = 10 \times 10 \times 10 = 1000$

$$n_2 \Rightarrow \left.\begin{array}{lll} i=1 & \Rightarrow & 1 \le j \le 8 \\ i=2 & \Rightarrow & 1 \le j \le 8 \\ i=3 & \Rightarrow & 2 \le j \le 8 \\ ; & & \\ ; & & \\ i=7 & \Rightarrow & j=8 \end{array}\right\}$$

$n_2 = 8 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 44$

$n_3 = {}^{10}C_4 . 4! = 210$

$n_4 = {}^{10}C_4 = 5040$

$\frac{n_4}{12} = 420$

ABD

2. Consider a triangle PQR having sides of lengths p, q and r opposite to the angles P, Q and R, respectively. Then which of the following statement is (are) **TRUE ?**

(A) $\cos P \ge 1 - \frac{p^2}{2qr}$

(B) $\cos R \ge \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)\cos Q$

(C) $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

(D) If $p < q$ and $p < r$, then $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

Ans. (A, B)

Sol. (A) $\cos P = \frac{q^2 + r^2 - p^2}{2qr}$

$\cos P = \frac{1}{2}\left(\frac{q}{r} + \frac{r}{q}\right) - \frac{p^2}{2qr}$ $\left(\frac{q}{r} + \frac{r}{q} \geq 2\right)$

$\cos P \geq 1 - \frac{p^2}{2qr}$

(B) Let $\cos R \geq \left(\frac{q-r}{p+q}\right)\cos P + \left(\frac{p-r}{p+q}\right)$ cos Q

(p + q) cos R ≥ (q – r) cos P + (p – r) cos Q

(P cos R + r cos P) + (q cos R + r cos Q) ≥ (q cos P + p cos Q)

q + p ≥ r (Triangle inequality)

(C) Let $\frac{q+r}{p} < 2\frac{\sqrt{\sin Q \sin R}}{\text{Sin } P}$

$\Rightarrow \frac{\sin Q + \sin R}{\sin P} < 2\frac{\sqrt{\sin Q \sin R}}{\sin P}$

$\Rightarrow \frac{\sin Q + \sin R}{2} < \sqrt{\sin Q \sin R}$

AM < GM (Not possible)

(D) Let $\cos Q > \frac{p}{r}$ and $\cos R > \frac{p}{q}$

$\frac{r^2 + p^2 - q^2}{2rp} > \frac{p}{r}$ and $\frac{p^2 + q^2 - r^2}{2pq} > \frac{p}{q}$

$\Rightarrow r^2 > p^2 + q^2$ and $q^2 > p^2 + r^2$ [Not possible]

3. Let $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}$ be a continuous function such that f(0) = 1 and $\int_0^{\frac{\pi}{3}} f(t)dt = 0$. The which of the following statements is (are) **TRUE ?**

(A) The equation f(x) – 3 cos 3x = 0 has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(B) The equation f(x) – 3 sin 3x = $-\frac{6}{\pi}$ has at least one solution in $\left(0, \frac{\pi}{3}\right)$

(C) $\lim\limits_{x\to 0}\dfrac{x\int_0^x f(t)dt}{1-e^{x^2}}$

(D) $\lim\limits_{x\to 0}\dfrac{\sin x\int_0^x f(t)dt}{x^2}=-1$

Ans. (A, B, C)

Sol. (A) $\int_0^{\frac{\pi}{3}}\left(f(x)-3\cos 3x\right)dx=\int_0^{\frac{\pi}{3}} f(x)dx-\int_0^{\frac{\pi}{3}} 3\cos 3x\,dx=0-\sin 3x\Big|_0^{\pi/3}=0$

$\Rightarrow$ f(x) – 3 cos 3x = 0 has atleast one solution in $\left(0,\dfrac{\pi}{3}\right)$

(B) $\int_0^{\frac{\pi}{3}}\left(f(x)-3\sin 3x+\dfrac{6}{\pi}\right)dx=\int_0^{\frac{\pi}{3}} f(x)dx-\int_0^{\frac{\pi}{3}} 3\sin 3x dx+\int_0^{\frac{\pi}{3}}\dfrac{6}{\pi}dx=0+\cos 3x\Big|_0^{\pi/3}+\dfrac{6}{\pi}.\dfrac{\pi}{3}=0$

$\Rightarrow$ f(x) – 3 sin 3x $+\dfrac{6}{\pi}=0$ has atleast one solution in $\left(0,\dfrac{\pi}{3}\right)$.

(C) $\lim\limits_{x\to\infty}\dfrac{x\int_0^x f(t)dt}{1-e^{x^2}}\left(\dfrac{0}{0}\right)$

Use L'hospital Rule

$$\lim_{x\to 0}\frac{x.f(x)+\int_0^x f(t)dt}{-2xe^{x^2}}\left(\frac{0}{0}\right)$$

Use L'Hospital Rule again

$$\lim_{x\to 0}\frac{x.f'(x)+2f(x)}{-2e^{x^2}-4x^2.e^{x^2}}=\frac{2.f(0)}{-2}=-1$$

(D) $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right).\dfrac{\int_0^x f(t)dt}{x}$

$$\lim_{x\to 0}\frac{\int_0^x f(t)dt}{x}\left(\frac{0}{0}\right)$$

Use L'Haspital Rule

$\lim_{x \to 0} \frac{f(x)}{1} = f(0) = 1$

As ABC

4. For any real numbers α and β let $y_{\alpha,\beta}$, $x \in \mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx} + \alpha y = xe^{\beta x}$, $y(1) = 1$. Let $S = \{y_{\alpha,\beta}(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S ?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(B) $f(x) = -\frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

Ans. (A, C)

Sol. $\frac{dy}{dx} + \alpha y = xe^{\beta x}$

I.F. = $e^{\int \alpha dx} = e^{\alpha x}$

$y.e^{\alpha x} = \int e^{\alpha x}.\left(xe^{\beta x}\right)dx + C$

$y.e^{\alpha x} = \int \underset{I}{x}.\underset{II}{e^{(\alpha+\beta)x}}\,dx + C$

$y.e^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{(\alpha+\beta)} - \int \frac{e^{(\alpha+\beta)x}}{\alpha+\beta}dx + C$

$y.e^{\alpha x} = e^{(\alpha+\beta)x}\left[\frac{x}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right] + C$

$y(1) = 1$

$\Rightarrow e^{\alpha} = e^{\alpha+\beta}\left[\frac{1}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right] + C$

$C = \left(1 - e^{\beta}\left(\frac{1}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right)\right)e^{\alpha}$

$y.e^{\alpha x} = e^{(\alpha+\beta)x}\left[\frac{x}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right] + e^{\alpha}\left(1 - e^{\beta}\left(\frac{1}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right)\right)$

$$y = e^{\beta x}\left(\frac{x}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right) + \left(e^{\alpha} - e^{\alpha+\beta}\left(\frac{1}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2}\right)\right)e^{-2x}$$

for $\alpha = \beta = 1$

$$y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$$

for $\alpha = 1, \beta = -1$

$$\frac{dy}{dx} + y = x.e^{-x}$$

IF $e^{\int dx} = e^x$

$$y.e^x = \int x\,dx = \frac{x^2}{2} + C$$

$y(1) = 1$

$$e = \frac{1}{2} + C \Rightarrow C = e - \frac{1}{2}$$

$$y.e^x = \frac{x^2}{2} + e - \frac{1}{2}$$

$$y = \frac{x^2}{2}.e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$$

As (A, C)

5. Let O be the origin and $\overrightarrow{OA} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\overrightarrow{OB} = \hat{i} - 2\hat{j} + 2\hat{k}$ and $\overrightarrow{OC} = \frac{1}{2}\left(\overrightarrow{OB} - \lambda\,\overrightarrow{OA}\right)$ for some $\lambda > 0$.

If $\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{9}{2}$, then which of the following statements is (are) **TRUE ?**

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$ is $-\frac{3}{2}$

(B) Area of the triangle OAB is $\frac{9}{2}$

(C) Area of the triangle ABC is $\frac{9}{2}$

(D) The acute angle between the diagonals of the parallelogram with adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{OC}$ is $\frac{\pi}{3}$

Ans. (A, B, C)

Sol. Let $\overrightarrow{OA} = \vec{a} = 2i + 2j + k$

$\overrightarrow{OB} = \vec{b} = i - 2j + 2k$

$\overrightarrow{OC} = \vec{c} = \frac{1}{2}\left(\vec{b} - \lambda\vec{a}\right)$

$\left|\overrightarrow{OB} \times \overrightarrow{OC}\right| = \frac{9}{2}$

$\left|\vec{b} \times \frac{\left(\vec{b} - \lambda\vec{a}\right)}{2}\right| = \frac{9}{2}$

$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix}$

$\frac{|\lambda|}{2}\left|\vec{b} \times \vec{a}\right| = \frac{9}{2}$

$\vec{a} \times \vec{b} = 6i - 3j - 6k$

$\frac{|\lambda|}{2}.9 = \frac{9}{2} \Rightarrow \lambda \pm 1 (\lambda > 0)$

$\vec{a} \times \vec{b} = 3(2i - j - 2k)$

$\lambda = 1$

$\vec{c} = \frac{\vec{b} - \vec{a}}{2}$

(A) Projection of $\overrightarrow{OC}$ on $\overrightarrow{OA}$

$= \frac{\vec{c}.\vec{a}}{|a|} = \frac{\left(\vec{b} - \vec{a}\right).\vec{a}}{2|\vec{a}|} = \frac{0 - |\vec{a}|^2}{2|\vec{a}|} = \frac{-3}{2}$

(B) Area of $\Delta OAB = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{9}{2}$

(C) Area of $\Delta ABC = \frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|$

$= \frac{1}{2}\left|\vec{a} \times \vec{b} + \vec{b} \times \left(\frac{\vec{b} - \vec{a}}{2}\right) + \left(\frac{\vec{b} - \vec{a}}{2}\right) \times \vec{a}\right|$

$= \frac{1}{2}\left|\vec{a} \times \vec{b}\right| = \frac{9}{2}$

(D) Let $\vec{d}_1 = \frac{\vec{a} + \vec{c}}{2} = \frac{\vec{a} + \left(\frac{\vec{b} - \vec{a}}{2}\right)}{2} = \frac{\vec{a} + \vec{b}}{4} = \frac{3i + 3k}{4}$

$$\vec{d}_2 = \frac{\vec{a}-\vec{c}}{2} = \frac{\vec{a}-\left(\frac{\vec{b}-\vec{a}}{2}\right)}{2} = \frac{3\vec{a}-\vec{b}}{4} = \frac{5i+8j+k}{4}$$

$$\cos\theta = \left|\frac{\vec{d}_1.\vec{d}_2}{|\vec{d}_1||\vec{d}_2|}\right|$$

$$\cos\theta = \frac{1}{\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

As (ABC)

6. Let E denote the parabola $y^2 = 8x$. Let P = (– 2, 4), and let Q and Q' be two distinct points on E such that the lines PQ and PQ' are tangents to E. Let F be the focus of E. Then which of the following statements is (are) **TRUE ?**

(1) The triangle PFQ is a right-angled triangle

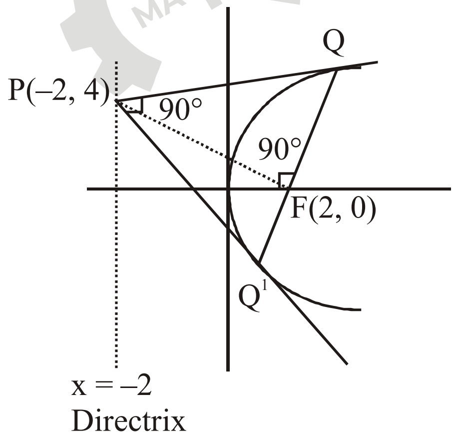
(2) The triangle QPQ' is a right-angled triangle

(3) The distance between P and F is $5\sqrt{2}$

(4) F lies on the line joining Q and Q'

Ans. (A, B, D)

Sol.



P lies on the directrix of the parabola.

⇒ QQ' is focal chord.

⇒ ∠QPQ' = 90° (Directrix is also director circle of the parabola)

$\Rightarrow \angle PFQ = 90°$ (Portion of tangent intercepted between point of contact and directrix subtends right angle at focus.)

$\Rightarrow$ $PF = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

As (ABD)

SECTION 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

  Zero Marks : 0 In all other cases.

Question Stem for Question Nos. 7 and 8

Question Stem

Consider the region $R = \{(x,y) \in \mathbb{R}\times\mathbb{R} : x \geq 0 \text{ and } y^2 \leq 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

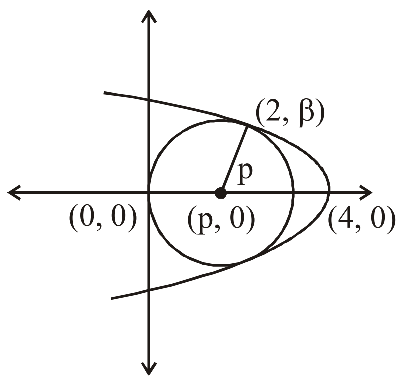
7. The radius of the circle C is ___ .

Ans. (1.5)

8. The value of α is ___ .

Ans. (2)

Sol.



$C : x^2 + y^2 - 2px = 0$

$P : y^2 = 4 - x$

By solving both the curves

$x^2 - (2p + 1)x + 4 = 0$

D = 0 (Both curves touch each other)

$(2p + 1)^2 - 4^2 = 0$

$(2p + 5)(2p - 3) = 0 \Rightarrow p = 3/2$ = radius of circle

for $p = \dfrac{3}{2} \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow x = 2 = \alpha$

Question Stem for Question Nos. 9 and 10

Question Stem

Let $f_1 : (0, \infty) \to R$ and $f_2 : (0, \infty) \to R$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t - j)^j \, dt, \quad x > 0$$

and $f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} + 2450, \ x > 0,$

where, for any positive integer n and real numbers $a_1, a_2, \ldots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \ldots, a_n$.

Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, i = 1, 2, in the interval $(0, \infty)$.

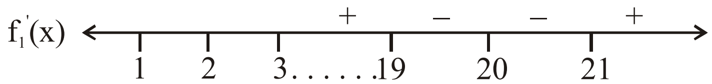
9. The value of $2m_1 + 3n_1 + m_1n_1$ is_____.

Ans. (57)

10. The value of $6m_2 + 4n_2 + 8m_2n_2$ is_____.

Ans. (6)

Sol. $f_1'(x) = \prod_{j=1}^{21} (x - j)^j = (x-1)^1 (x-2)^2 (x-3)^3 \ldots\ldots (x-21)^{21}$



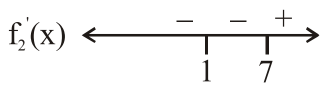
Points of minima $\Rightarrow x = 21, 17, 13, 9, 5, 1 \Rightarrow m_1 = 6$

Points of maxima $\Rightarrow x = 19, 15, 11, 7, 3 \Rightarrow n_1 = 5$

$2m_1 + 3n_1 + m_1n_1 = 57.$

$f_2'(x) = 98.50(x-1)^{49} - 600.49(x-1)^{48}$

$f_2'(x) = 4900(x-1)^{48}(x-7)$



Points of minima $\Rightarrow x = 7$ $\qquad m_2 = 1$

$n_2 = 0$

$6m_2 + 4n_2 + 8m_2n_2 = 6.$

Question Stem for Question Nos. 11 and 12

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R, i = 1, 2,$ and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$ be functions such that

$g_1(x) = 1, g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x,$ for all $x \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$

Define $S_i = \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} f(x) \cdot g_i(x) dx, \quad i = 1, 2$

11. The value of $\frac{16S_1}{\pi}$ is____.

Ans. (2)

12. The value of $\frac{48S_2}{\pi^2}$ is______.

Ans. (1.5)

Sol. $S_1 = \int_{\pi/8}^{3\pi/8} f(x).1dx = \int_{\pi/8}^{3\pi/8} \sin^2 x dx$ ....(1)

apply $\int_a^b f(a + b - x) dx = \int_a^b f(x) dx$

$S_1 = \int_{\pi/8}^{3\pi/8} \cos^2 x dx$ .... (2)

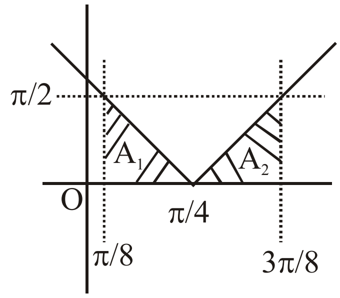
$(1) + (2) \Rightarrow 2S_1 = \int_{\pi/8}^{3\pi/8} dx = \frac{\pi}{4} \Rightarrow S_1 = \frac{\pi}{8}$

$\frac{16S_1}{\pi} = 2$

$S_2 = \int_{\pi/8}^{3\pi/8} \sin^2 x |4x - \pi| dx$ ......(3)

apply $\int_a^b f(a + b - x) dx = \int_a^b f(x) dx$

$$S_2 = \int_{\pi/8}^{3\pi/8} \cos^2 x\,|4x-\pi|\,dx \quad ....(4)$$



$(3) + (4) \Rightarrow 2S_2 = \int_{\pi/8}^{3\pi/8} |4x-\pi|\,dx = \frac{\pi^2}{16}$

$A_1 + A_2 = \frac{1}{2}\times\frac{\pi}{4}\times\frac{\pi}{2} = \frac{\pi^2}{16}$

$\Rightarrow S_2 = \frac{\pi^2}{32}$

$\Rightarrow \frac{48}{\pi^2}S_2 = 1.5$

SECTION 3

- This section contains **TWO (02)** paragraphs. Based on each paragraph, there are **TWO (02)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : -1 In all other cases.

Question Stem for Question Nos. 13 and 14

Let $M = \{(x,y) \in R\times R : x^2 + y^2 \le r^2\}$,

where r > 0. Consider the geometric progression $a_n = \frac{1}{2^{n-1}}, n = 1, 2, 3, ...$ Let $S_0 = 0$ and, for $n \ge 1$, let S_n denote the sum of the first n terms of this progression. For $n \ge 1$, let C_n denote the circle with center $(S_{n-1}, 0)$ and radius a_n, and D_n denote the circle with center (S_{n-1}, S_{n-1}) and radius a_n.

13. Consider M with $r = \frac{1025}{512}$. Let k be the number of all those circles C_n that are inside M. Let l be the maximum possible number of circles among these k circles such that no two circles intersect. Then

(1) $k + 2l = 22$ (2) $2k + l = 26$ (3) $2k + 3l = 34$ (4) $3k + 2l = 40$

Ans. (4)

Sol. $a_n = \frac{1}{2^{n-1}}$

$S_n = 1 + \frac{1}{2} + ...\frac{1}{2^{n-1}}$

$S_n = 2 - \frac{1}{2^{n-1}}$

hence centre of C_n is $\left(2 - \frac{1}{2^{n-2}}, 0\right)$ and radius of C_n is $\frac{1}{2^{n-1}}$

$r = \frac{1025}{513} < 2$

here C_n inside M then

$d < |r_1 - r_2|$

$2 - \frac{1}{2^{n-2}} < 2 - \frac{1}{2^{n-1}}$

$2 - \frac{1}{2^{n-2}} + \frac{1}{2^{n-1}} < 2$

$\Rightarrow K = 10$

also we observe that $l = 5$ then option (D) is correct.

14. Consider M with $r = \frac{(2^{199}-1)\sqrt{2}}{2^{198}}$. The number of all those circles D_n that are inside M is

(1) 198 (2) 199 (3) 200 (4) 201

Ans. (2)

Sol. Centre of D_n here $S_n = 2 - \frac{1}{2^{n-1}}$

(S_{n-1}, S_{n-1}) and radius is $\frac{1}{2^{n-1}}$

D_n lies inside M then

$d < |r_1 - r_2|$

$\sqrt{2}\,S_{n-1} < \frac{(2^{199}-1)\sqrt{2}}{2^{198}} - \frac{1}{2^{n-1}}$

$\sqrt{2}\left(2 - \frac{1}{2^{n-2}}\right) < \frac{(2^{199}-1)\sqrt{2}}{2^{198}} - \frac{1}{2^{n-1}}$

$\frac{\sqrt{2}}{2^{n-2}} > \frac{\sqrt{2}}{2^{198}} + \frac{1}{2^{n-1}}$

$\Rightarrow$ n = 199.

Question Stem for Question Nos. 15 and 16

Let $\psi_1 : [0,\infty) \to R, \psi_2 : [0,\infty) \to R, f : [0,\infty) \to R$ and $g : [0,\infty) \to R$ be functions such that $f(0) = g(0) = 0,$

$\psi_1(x) = e^{-x} + x, \quad x \ge 0,$

$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2, \quad x \ge 0,$

$f(x) = \int_{-x}^{x} (|t| - t^2)e^{-t^2}dt, \quad x > 0$

and $g(x) = \int_0^{x^2} \sqrt{t}e^{-t}dt, \quad x > 0.$

15. Which of the following statements is TRUE ?

(1) $f(\sqrt{\ln 3}) + g(\sqrt{\ln 3}) = \frac{1}{3}$

(2) For every x > 1, there exists an $\alpha \in (1, x)$ such that $\psi_1(x) = 1 + \alpha x$

(3) For every x > 0, there exists a $\beta \in (0, x)$ such that $\psi_2(x) = 2x(\psi_1(\beta) - 1)$

(4) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

Ans. (3)

Sol. $f(x) = 2\int_0^x \left(t - t^2\right)e^{-t^2}dt$

$f'(x) = 2\left(x - x^2\right)e^{-x^2}$

$g'(x) = 2x^2e^{-x^2}$

Option (A)

Let consider

$f'(x) + g'(x) = 2\left(x - x^2\right)e^{-x^2} + 2x^2e^{-x^2}$

Integrating both side

$f(x) + g(x) = -e^{-x^2} + c$

put x = 0

then c = 1

$f(x) + g(x) = 1 - e^{-x^2}$

$f(\sqrt{\ln 3})+g(\sqrt{\ln 3})=1-e^{-\ln 3}=1-\frac{1}{3}=\frac{2}{3}$

hence option (A) is wrong.
Option (B)
Let consider

$H(x)=\psi_1(x)-1-\alpha x$

$H(1)=\psi_1(1)-1-\alpha$

$H(1)=e^{-1}+1-1-\alpha$

$H(1)=e^{-1}-\alpha<0$

$H'(x)=\psi_1'(x)-\alpha$

$H'(x)=-e^{-x}+1-\alpha<0$

hence H(x) is decresing.
So H(x) is always negative and $H(x)\neq 0$
Hence option (B) is wrong.
Option (C)

$\psi_2(x)=x^2-2x-2e^{-x}+2$

Applying LMVT in (0, x)

$\psi'_2(\beta)=\frac{\psi_2(x)-\psi_2(0)}{x-0}$

$\psi'_2(\beta)=\frac{\psi_2(x)-0}{x}$ ....(1)

$\psi'_2(x)=2x-2+2e^{-x}$

$\psi'_2(\beta)=2\beta-2+2e^{-\beta}$

$\psi'_2(\beta)=2\left(\beta+e^{-\beta}\right)-2$

$\psi'_2(\beta)=2\psi_1(\beta)-2$

Now by (1)

$2(\psi_1(\beta)-1)=\frac{\psi_2(x)}{x}$

$\psi_2(x)=2x(\psi_1(\beta)-1)$

hence option (C) is correct.
Option (D)

$f(x)=\int_{-x}^{x}(|t|)-t^2)e^{-t^2}\,dt$

$f(x)=2\int_{0}^{x}(t-t^2)e^{-t^2}dt$

$f'(x) = 2(x - x^2)e^{-x^2}$

$f'(x) = 2x(1 - x)\,e^{-x^2}$

$- - - - - -$ $+ + + + + + +$ $- - - - - -$

0   1

So option (D) is wrong.

16. Which of the following statements is TRUE ?

(1) $\psi_1(x) \le 1,$ for all x > 0

(2) $\psi_2(x) \le 0,$ for all x > 0

(3) $f(x) \ge 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5,$ for all $x \in \left(0, \frac{1}{2}\right)$

(4) $g(x) \le \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7,$ for all $x \in \left(0, \frac{1}{2}\right)$

Ans. (4)

Sol. Option (A)

$\psi_1(x) = e^{-x} + x$

$\psi'_1(x) = -e^{-x} + 1$

$\psi'_1(x) > 0$

hence $\psi_1(x)$ is increasing

then

$\psi_1(x) \ge \psi_1(0)$

$\psi_1(x) \ge 1$

So option (A) is wrong.

Option (B)

$\psi_2(x) = x^2 - 2x - 2e^{-x} + 2$

$\psi_2'(x) = 2x - 2 + 2e^{-x}$

$\psi_2'(x) > 0$

hence $\psi_2'(x)$ is incresing.

then

$\psi_2(x) \ge \psi_2(0)$

$\psi_2(x) \ge 0$

hence option (B) is wrong.

Option (C)

$f(x) \geq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$

Consider

$H(x) = f(x) - 1 + e^{-x^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5$

$H'(x) = f'(x) - 2xe^{-x^2} + 2x^2 - 2x^4$

but $f'(x) = 2(x - x^2)e^{-x^2}$

Now

$H'(x) = 2x^2(1 - x^2 - e^{-x^2})$

$H'(x) = 2x^2(1 - x^2 - 1 + \frac{x^2}{1} - \frac{x^4}{2} + ...) < 0$

then H(x) is decreasing

$H(x) \leq H(0)$

$H(x) \leq f(0)$

$H(x) \leq 0$

hence

$f(x) \leq 1 - e^{-x^2} - \frac{2}{3}x^3 + \frac{2}{5}x^5$

hence option (C) is wrong.

Option (D)

$g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$

Let consider

$H(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7$

$H'(x) = g'(x) - 2x^2 + 2x^4 - x^6$

but $g'(x) = 2x^2e^{-x^2}$

$H'(x) = 2x^2e^{-x^2} - 2x^2 + 2x^4 - x^6$

$H'(x) = 2x^2\left(1 - \frac{x^2}{1} + \frac{x^4}{2} - \frac{x^6}{3}.......\right) - 2x^2 + 2x^4 - x^6 < 0$

H(x) is decreasing.

So

$H(x) \leq H(0)$

$g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7 \leq 0$

$g(x) \leq \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7$

hence option (D) is correct.

SECTION 4

- This section contains **THREE (03)** questions.
- The answer to each question is a **NON-NEGATIVE INTEGER**.
- For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If ONLY the correct integer is entered;

  Zero Marks : 0 In all other cases.

17. A number is chosen at random from the set {1,2,3, ..., 2000}. Let p be the probability that the chosen number is a multiple of 3 or a multiple of 7. Then the value of 500p is____.

Ans. (214)

Sol. Number divisible by '3' in this set = 3, 6, 9, ...1998
total such numbers = 666
Numbers divisible by '7' in this set = 7, 14, ...1995
total such numbers = 285
total numbers divisible by '3' or '7' = (666 + 285) – (numbers divisible by '21')
Numbers divisible by '21' in this set = 21, 42, ... 1995
total such numbers = 95
Now numbers divisible by '3' or '7' = 666 + 285 – 95 = 856.

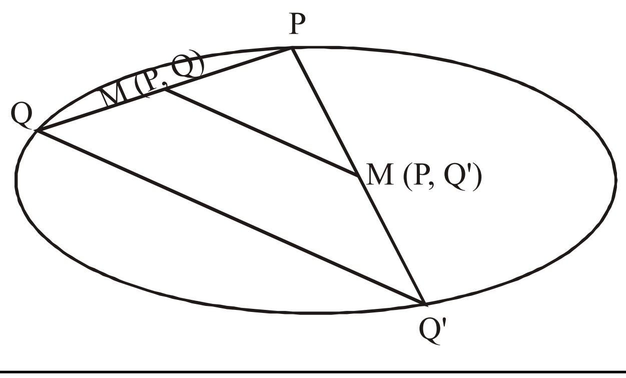
$$P = \frac{856}{2000}$$

Now, 500P = 214.

18. Let E be the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. For any three distinct points P, Q and Q' On E, let M (P, Q) be the mid-point of the line segment joining P and Q, and M (P, Q') be the mid-point of the line segment joining P and Q'. Then the maximum possible value of the distance between M (P, Q) and M(P, Q'), as P, Q and Q' vary on E, is_____.

Ans. (4)

Sol.



We know from triangle geometry

distance between M(P, Q) and M(P, Q') = $\frac{1}{2}$QQ'

hence distance between M(P, Q) and M(P, Q') will be maximum if QQ' is maximum and in given ellipse maximum value of QQ' will be 2a

then maximum distance between M(P, Q) and M(P, Q') = $\frac{1}{2}$ (2a) = $\frac{8}{2}$ = 4

19. For any real number x, let [x] denote the largest integer less than or equal to x. If $I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx$, then the value of 9I is_____.

Ans. (182)

Sol. $f(x) = \frac{10x}{x+1}$

$f'(x) = \frac{10x}{(x+1)^2} > 0$

hence f'(x) is increasing
Now

$$I = \int_0^{\frac{1}{9}} 0dx + \int_{\frac{1}{9}}^{\frac{2}{3}} 1dx + \int_{\frac{2}{3}}^{9} 2dx + \int_9^{10} 3dx = \frac{182}{9}$$

Now 9I = 182
Ans = 182