JEE Adv. October 2021

Question Paper With Text Solution

03 October. | Paper-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

Office : Piprali Road, Sikar (Raj.) | Ph. 01572-241911

Website : www.matrixedu.in ; Email : smd@matrixacademy.co.in

JEE ADV. OCTOBER 2021 | 03 OCTOBER PAPER-1

SECTION – A

- This section contains **FOUR (04)** questions.
- Each question has **FOUR** options (A), (B), (C) and (D). **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +3 If ONLY the correct option is chosen;

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered);

Negative Marks : –1 In all other cases.

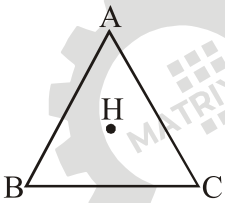
1. Consider a triangle Δ whose two sides lie on the x-axis and the line $x + y + 1 = 0$. If the orthocenter of Δ is (1, 1), then the equation of the circle passing through the vertices of the triangle Δ is

(A) $x^2 + y^2 - 3x + y = 0$ (B) $x^2 + y^2 + x + 3y = 0$

(C) $x^2 + y^2 + 2y - 1 = 0$ (D) $x^2 + y^2 + x + y = 0$

Ans. (B)

Sol.



equation of AC $x + y + 1 = 0$

equation of BC $y = 0$

orthocenter H(1, 1)

equation of AH $x = 1$

equation of BH $x - y = 0$

Get A = (1, –2); B(0, 0) C(–1, 0)

Let circum center P(h, k)

PA = PB PB = PC

$(h - 1)^2 + (k + 2)^2 = h^2 + k^2$ $h^2 + k^2 = (h + 1)^2 + k^2$

$\Rightarrow h = \frac{-1}{2}$; $k = -\frac{3}{2}$

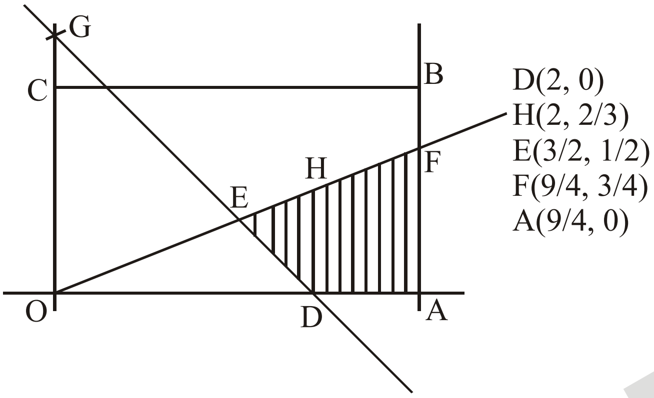
equation of circum circle $x^2 + y^2 + x + 3y = 0$

2. The area of the region $\left\{(x,y): 0 \le x \le \frac{9}{4},\quad 0 \le y \le 1,\quad x \ge 3y,\quad x+y \ge 2\right\}$ is

(A) $\frac{11}{32}$ (B) $\frac{35}{96}$ (C) $\frac{37}{96}$ (D) $\frac{13}{32}$

Ans. (A)

Sol.



$AB \equiv x = \frac{9}{4}$

$BC \equiv y = 1$

$DG \equiv x + y = 2$

$OF \equiv x = 3y$

$DG \equiv x = 2$

we will divide shaded region in two parts

required area = Area of ΔDEH + area of trapezium AFHD

$= \frac{1}{2}\left(\frac{2}{3}\right)\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{2}{3}+\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{11}{32}$

3. Consider three sets $E_1 = \{1,2,3\}$ $F_1 = \{1,3,4\}$ and $G_1 = \{2,3,4,5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 donote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 donote the set of these chosen elements.

Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements.

Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1,2\}$. Then the value of p is

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$ (C) $\frac{1}{2}$ (D) $\frac{2}{5}$

Ans. (A)

Sol. $E_3 = E_1 \Rightarrow S_3 = S_1$

$E_1\{1, 2, 3\}$ →

- $S_1\{1, 2\} \Rightarrow F_2 = \{1, 2, 3, 4\}$
  - $S_2 = \{2, 3\}, \{2, 4\}, \{3, 4\} \to X$
  - $S_2 = \{1, 2\}$ or $\{1, 3\}$ or $\{1, 4\} \Rightarrow G_2 = \{1, 2, 3, 4, 5\} \to S_3 = \{1, 2\} P_1$
- $S_1\{2, 3\} \Rightarrow F_2 = \{1, 2, 3, 4\}$
  - $S_2 = \{2, 3\}$ or $\{2, 4\}$ or $\{3, 4\} \to G_2 = \{2, 3, 4, 5\} \to S_3 = \{2, 3\} P_2$
  - $S_2 = \{1, 2\}$ or $\{1, 3\}$ or $\{1, 4\} \Rightarrow G_2 = \{1, 2, 3, 4, 5\} \to S_3 = \{2, 3\} P_3$
- $S_1\{1, 3\} \Rightarrow F_2 = \{1, 3, 4\}$
  - $S_2 = \{1, 3\}$ or $\{1, 4\} \Rightarrow G_2 = \{1, 2, 3, 4, 5\} \to S_3 = \{1, 3\} P_4$
  - $S_2 = \{3, 4\} \Rightarrow X$

P_i denotes probability of respective branch

Required probability = $\dfrac{P_1}{P_1 + P_2 + P_3 + P_4} = \dfrac{\left(\frac{1}{3}\times\frac{3}{6}\times\frac{1}{10}\right)}{\left(\frac{1}{3}\times\frac{3}{6}\times\frac{1}{10}\right)+\left(\frac{1}{3}\times\frac{3}{6}\times\frac{1}{6}\right)+\left(\frac{1}{3}\times\frac{3}{6}\times\frac{1}{10}\right)+\left(\frac{1}{3}\times\frac{2}{3}\times\frac{1}{10}\right)}$

$= \dfrac{3}{3+5+3+4} = \dfrac{3}{15} = \dfrac{1}{5}$

4. Let $\theta_1, \theta_2, ..., \theta_{10}$ be positive valued angles (in radian) such that $\theta_1 + \theta_2 + ... + \theta_{10} = 2\pi$. Define the complex numbers $z_1 = e^{i\theta_1}$, $z_k = z_{k-1}e^{i\theta_k}$ for k = 2,3, ..., 10, where $i = \sqrt{-1}$. Consider the statements P and Q given below:

$P: |z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}| \le 2\pi$

$Q: |z_2^2 - z_1^2| + |z_3^2 - z_2^2| + ... + |z_{10}^2 - z_9^2| + |z_1^2 - z_{10}^2| \le 4\pi$

Then,

(A) P is TRUE and Q is FALSE
(B) Q is TRUE and P is FALSE
(C) both P and Q are TRUE
(D) both P and Q are FALSE

Ans. (C)

Sol. $\theta_1 + \theta_2 + ... \theta_{10} = 2\pi$

$z_1 = e^{i\theta_1}$, $z_2 = z_1 e^{i\theta_2}$, $z_3 = z_2 e^{i\theta_3}$ ....

$|z_k - z_{k-1}| = |z_{k-1}e^{i\theta_k} - z_{k-1}|$

$= |z_{k-1}||e^{i\theta_k} - 1| = \left|2\sin\left(\frac{\theta_k}{2}\right)\right|$

$$\left|z_k^2 - z_{k-1}^2\right| = \left|z_{k-1}^2 e^{i2\theta_k} - z_{k-1}^2\right| = \left|z_{k-1}^2\right|\left|e^{i2\theta_{k-1}}\right| = \left|2\sin\theta_k\right|$$

Statement P

$$|z_2 - z_1| + |z_3 - z_2| + ... + |z_{10} - z_9| + |z_1 - z_{10}|$$

$$\left|2\sin\frac{\theta_2}{2}\right| + \left|2\sin\frac{\theta_3}{2}\right| + ... + \left|2\sin\frac{\theta_{10}}{2}\right| + \left|2\sin\left(\frac{\theta_2 + ... + \theta_{10}}{2}\right)\right|$$

$$\left|2\sin\frac{\theta_2}{2}\right| + \left|2\sin\frac{\theta_3}{2}\right| + ... + \left|2\sin\frac{\theta_{10}}{2}\right| + \left|2\sin\left(\frac{2\pi - \theta_1}{2}\right)\right|$$

$$= \left|2\sin\frac{\theta_1}{2}\right| + \left|2\sin\frac{\theta_2}{2}\right| + ... + \left|2\sin\frac{\theta_{10}}{2}\right|$$

Maximum value will occur when $\theta_1 = \theta_2 ... = \theta_{10} = \frac{2\pi}{10}$

$$\text{LHS}_{\max} = 10\left(2\sin\frac{\pi}{10}\right) = 20\sin(18°) = 20\left(\frac{\sqrt{5}-1}{4}\right) \approx 6.18 < 2\pi$$

$\Rightarrow$ P is true

Statement Q

LHS = $|2 \sin \theta_1| + |2 \sin \theta_2| + ... + |2 \sin \theta_{10}|$

Maximum Value will occur when $\theta_1 = \theta_2 = ... \theta_{10} = \frac{\pi}{5}$.

$$\text{LHS} = 10\left(2\,\text{Sin}\frac{\pi}{5}\right) = 20\,\text{Sin}\,36° < 4\pi$$

$\Rightarrow$ Statement Q is true.

SECTION 2

- This section contains **THREE (03)** question stems.
- There are **TWO (02)** questions corresponding to each question stem.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value corresponding to the answer in the designated place using the mouse and the on-screen virtual numeric keypad.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +2 If ONLY the correct numerical value is entered at the designated place;

Zero Marks : 0 In all other cases.

5. Three numbers are chosen at random, one after another with replacement, from the set $S = \{1,2,3,...,100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

The value of $\frac{625}{4}p_1$ is____.

Ans. (76.25)

Sol. $S \equiv \{1, 2, 3, ...100\}$

For p_1

we will use complementary approach

$p_1 = 1 - p$ (All elements are less than 81)

$$p_1 = 1 - \left(\frac{80}{100}\right)^3 = 1 - \left(\frac{4}{5}\right)^3 = \frac{61}{125}$$

$$\frac{625p_1}{4} = \frac{625}{4} \times \frac{61}{125} = 76.25$$

6. Three numbers are chosen at random, one after another with replacement, from the set $S = \{1,2,3,...,100\}$. Let p_1 be the probability that the maximum of chosen numbers is at least 81 and p_2 be the probability that the minimum of chosen numbers is at most 40.

The value of $\frac{125}{4}p_2$ is____.

Ans. (24.5)

Sol. For p_2 we will use complamentary aproach

$p_2 = 1 - p$ (All element are greater than 40)

$$p_2 = 1 - \left(\frac{60}{100}\right)^3 = 1 - \left(\frac{3}{5}\right)^3 = \frac{98}{125}$$

$$\frac{125}{4}p_2 = \frac{125}{4} \times \frac{98}{125} = 24.5$$

7. Let α, β and γ be real numbers such that the system of linear equations

$x + 2y + 3z = \alpha$

$4x + 5y + 6z = \beta$

$7x + 8y + 9z = \gamma - 1$

is consistent. Let |M| represent the determinant of the matrixs

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the square of the distance of the point (0,1,0) from the plane P.

The value of |M| is_____.

Ans. (1)

Sol. $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

$\Rightarrow \Delta_x = 0$

$$\Rightarrow \begin{vmatrix} \alpha & 2 & 4 \\ \beta & 1 & 0 \\ \gamma-1 & 8 & 9 \end{vmatrix} = 0$$

$$|M| = \begin{vmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$|M| = \alpha - 2\beta + \gamma = 1$

8. Let α, β and γ be real numbers such that the system of linear equations

$x + 2y + 3z = \alpha$

$4x + 5y + 6z = \beta$

$7x + 8y + 9z = \gamma - 1$

is consistent. Let |M| represent the determinant of the matrixs

$$M = \begin{bmatrix} \alpha & 2 & \gamma \\ \beta & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Let P be the plane containing all those (α, β, γ) for which the above system of linear equations is consistent, and D be the square of the distance of the point (0,1,0) from the plane P.

The value of D is_____.

Ans. (1.5)

Sol. Plane P $x - 2y + z = 1$

$$\sqrt{D} = \left|\frac{0-2+0-1}{\sqrt{1+4+1}}\right| = \frac{3}{\sqrt{6}}$$

$$D = \frac{9}{6} = 1.5$$

9. Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2} + y - 1 = 0$ and $L_2 : x\sqrt{2} - y + 1 = 0$

For a fixed constant λ let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line $y = 2x + 1$ meets C at two points R and S. where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'.

The value of λ^2 is_____.

Ans. (9)

Sol. Let P(x, y)

$d_1 d_2 = \lambda^2$

$$\left|\frac{x\sqrt{2}-y+1}{\sqrt{3}}\right|\left|\frac{x\sqrt{2}+y-1}{\sqrt{3}}\right| = \lambda^2$$

$$\left|2x^2 - (y-1)^2\right| = 3\lambda^2$$

we will shift origin to (0, 1)

$x - 0 = X$; $y - 1 = Y$

$|2X^2 - Y^2| = 3\lambda^2$ $\quad\quad$ $y = 2x + 1 \Rightarrow Y = 2X$

For R & S

Put $Y = 2X$ in equation of curve C

$|2X^2 - 4X^2| = 3\lambda^2$

$2X^2 = 3\lambda^2$

$$X = \pm\frac{\sqrt{3}}{\sqrt{2}}\lambda$$

$$R\left(\frac{\sqrt{3}}{\sqrt{2}}\lambda, \frac{2\sqrt{3}}{\sqrt{2}}\lambda\right) \quad\quad S\left(\frac{-\sqrt{3}}{\sqrt{2}}\lambda, \frac{-2\sqrt{3}}{\sqrt{2}}\lambda\right)$$

Midpoint of RS = (0, 0)

$$RS = \left|\sqrt{5}\times 2\left(\frac{\sqrt{3}}{\sqrt{2}}\right)\lambda\right| = \sqrt{270}$$

$30\lambda^2 = 270$

$\lambda^2 = 9$

10. Consider the lines L_1 and L_2 defined by

$L_1 : x\sqrt{2} + y - 1 = 0$ and $L_2 : x\sqrt{2} - y + 1 = 0$

For a fixed constant λ let C be the locus of a point P such that the product of the distance of P from L_1 and the distance of P from L_2 is λ^2. The line y = 2x + 1 meets C at two points R and S. where the distance between R and S is $\sqrt{270}$.

Let the perpendicular bisector of RS meet C at two distinct points R' and S'. Let D be the square of the distance between R' and S'.

The value of D is_____.

Ans. (77.14)

Sol. Perpendicular bisector of RS

= Line perpendicular to RS & passing through midpoint of RS = (0, 0)

X + 2Y = 0

for R' and S'

Put X = – 2Y in equation of curve C

$\left|8Y^2 - Y^2\right| = 3\lambda^2$

$7Y^2 = 3\lambda^2$

$7Y^2 = 27$

$Y^2 = \frac{27}{7}$

$R'\left(-\frac{2\sqrt{27}}{\sqrt{7}}, \sqrt{\frac{27}{7}}\right) \quad S'\left(\frac{2\sqrt{27}}{\sqrt{7}}, \frac{-\sqrt{27}}{\sqrt{7}}\right)$

$R'S' = 2\sqrt{5}\sqrt{\frac{27}{7}} = \sqrt{D}$

$D = 20\times\frac{27}{7} = 77.14$

SECTION 3

- This section contains SIX (06) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

  Full Marks : +4 If only (all) the correct option(s) is(are) chosen;

  Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

  Partial Marks : +2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

  Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option; Zero Marks : 0 If unanswered; Negative Marks : —2 In all other cases.
- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then choosing ONLY (A), (B) and (D) will get +4 marks; choosing ONLY (A) and (B) will get +2 marks; choosing ONLY (A) and (D) will get +2marks; choosing ONLY (B) and (D) will get +2 marks; choosing ONLY (A) will get +1 mark; choosing ONLY (B) will get +1 mark; choosing ONLY (D) will get +1 mark; choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get –2 marks.

11. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let

$$E=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 4\\ 8 & 13 & 18\end{bmatrix}, P=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix} \text{ and } F=\begin{bmatrix}1 & 3 & 2\\ 8 & 18 & 13\\ 2 & 4 & 3\end{bmatrix}$$

If Q is a nonsingular matrix of order 3 × 3, then which of the following statements is (are) TRUE ?

(A) F=PEP and $P^2=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

(B) $|EQ + PEQ^{-1}| = |EQ| + |PFQ^{-1}|$

(C) $|(EF)^3| > |EF|^2$

(D) Sum of the diagonal entries of $P^{-1}EP + F$ is equal to the sum of diagonal entries of $E + P^{-1}FP$

Ans. (A,B,D)

Sol. $E=\begin{bmatrix}1&2&3\\2&3&4\\8&13&18\end{bmatrix};P=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix};F=\begin{bmatrix}1&3&2\\8&18&13\\2&4&3\end{bmatrix}$

$|Q|\neq 0$, Q is square matrix of order 3

(A) $PE=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&2&3\\2&3&4\\8&13&18\end{bmatrix}=\begin{bmatrix}1&2&3\\8&13&18\\2&3&4\end{bmatrix}$

$$PEP=\begin{bmatrix}1&2&3\\8&13&18\\2&3&4\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&3&2\\8&18&13\\2&4&3\end{bmatrix}=F$$

$$P^2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$\Rightarrow$ (A) is Correct

(B) F = PEP & $P^2=I$ (from (A))

$PF = P^2EP$

PF = IEP

PF = EP -------(1)

LHS = $|EQ + PFQ^{-1}|$

$=|EQ + EPQ^{-1}|$

$|E||Q+PQ^{-1}|$

=0

RHS = $|EQ| + |PFQ^{-1}|$

$= |EQ| + |EPQ^{-1}|$

$=|E||Q| + |E|\,|PQ^{-1}|$

= 0

$\because |E|=0$

(C) |EF|=|E||F|=0

$\Rightarrow$ (C) is Incorrect

LHS :

(D) tr $(P^{-1}EP + F)$

= tr $(P^{-1}PF + F)$ (Using (1))

= tr (F + F)

= tr (2F) = 2 tr(F) = 2 (1 + 18 + 3) = 44

RHS :

tr$(E + P^{-1}FP)$

= tr $(E + P^{-1}EP)$ (Using (1))

Now, $\because P^2=I\Rightarrow P=P^{-1}$

= tr (E + PEP)

= tr (E) + tr (PEP)
= tr(E) + tr(F)
= (1 + 3 + 18) + (1 + 18 + 3)
= 44
$\Rightarrow$ (D) is correct

12. Let f : R $\rightarrow$ R be defined by

$f(x)=\frac{x^2-3x-6}{x^2+2x+4}$ Then which of the following statements is (are)TRUE ?

(A) f is decreasing in the interval (–2, –1)

(B) f is increasing in the interval (1, 2)

(C) f is onto

(D) Range of f is $\left[-\frac{3}{2},2\right]$

Ans. (A,B)

Sol. f: R $\rightarrow$ R

$$f(x)=\frac{x^2-3x-6}{x^2+2x+4}$$

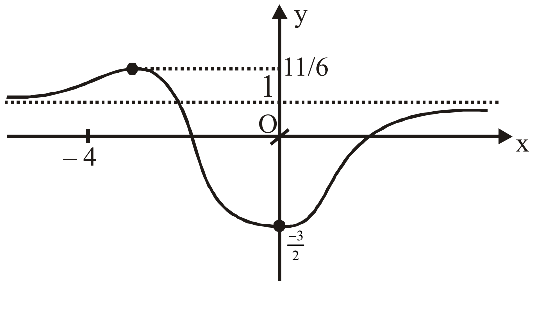
$$f'(x)=\frac{(2x-3)(x^2+2x+4)-(2x+2)(x^2-3x-6)}{(x^2+2x+4)^2}$$

$$f'(x)=\frac{5x(x+4)}{(x^2+2x+4)^2} \Rightarrow \quad \underset{-4}{+\,|\,-}\quad \underset{0}{|\,+}$$

f(x) is decreasing in (–2, –1) $\Rightarrow$ (A) is correct
f(x) is increasing in (1, 2) $\Rightarrow$ (B) is correct

Now, $f(-4)=\frac{16+12-6}{16-8+4}=\frac{11}{6}$; $f(0)=\frac{-3}{2}$

$\lim\limits_{x\to\infty} f(x) = 1$ & $\lim\limits_{x\to\infty} f(x) = 1$



Range is $\left[\frac{-3}{2}, \frac{11}{6}\right] \Rightarrow$ (D) is incorrect

Range $\neq$ Co-domain $\Rightarrow$ (C) is incorrect

13. Let E, F and G be three events having probabilities

$P(E) = \frac{1}{8}, P(F) = \frac{1}{6}$ and $P(G) = \frac{1}{4}$, and let $P(E \cap F \cap G) = \frac{1}{10}$

For any event H, if H^c denotes its complement, then which of the following statements is (are) TRUE ?

(A) $P(E \cap F \cap G^c) \leq \frac{1}{40}$

(B) $P(E^c \cap F \cap G) \leq \frac{1}{15}$

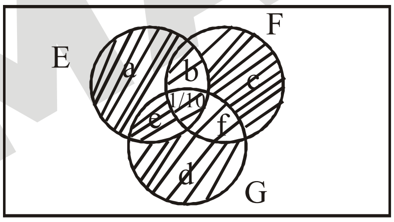
(C) $P(E \cup F \cup G) \leq \frac{13}{24}$

(D) $P(E^c \cap F^c \cap G^c) \leq \frac{15}{12}$

Ans. (A,B,C)

Sol. $P(E) = \frac{1}{8}$, $P(F) = \frac{1}{6}$ and $P(G) = \frac{1}{4}$

$P(E \cap F \cap G) = \frac{1}{10}$



$P(E) = \frac{1}{8} \Rightarrow a + b + \frac{1}{10} + e = \frac{1}{8} \Rightarrow a + b + e = \frac{1}{40}$ ....(1)

$P(F) = \frac{1}{6} \Rightarrow b + c + f + \frac{1}{10} = \frac{1}{6} \Rightarrow b + c + f = \frac{1}{15}$ ....(2)

$P(G) = \frac{1}{4} \Rightarrow e + d + f + \frac{1}{10} = \frac{1}{4} \Rightarrow e + d + f = \frac{3}{20}$ ....(3)

(A) $E \cap F \cap G^C = b$

from (1) & (2), $b \leq \frac{1}{15}$ & $b \leq \frac{1}{40}$ $\Rightarrow P\left(E \cap F \cap G^C\right) \leq \frac{1}{40}$

$\Rightarrow$ (A) is correct

(B) $E^C \cap F \cap G = f$

From (2) & (3) $f \leq \frac{1}{15}$ & $f \leq \frac{3}{20}$ $\Rightarrow P\left(E^C \cap F \cap G\right) \leq \frac{1}{15}$

$\Rightarrow$ (B) is correct

(C) $P(E\cup F\cup G)\le P(E)+P(F)+P(G)$

$\le \frac{1}{8}+\frac{1}{6}+\frac{1}{4}\le\frac{13}{24}$

$\Rightarrow$ (C) is correct

(D) $P\left(E^C\cap F^C\cap G^C\right)=1-P(E\cup F\cup G)$

$P(E\cup F\cup G)\le\frac{13}{24}$ (from (C))

$1-P(E\cup F\cup G)\ge 1-\frac{13}{24}\ge\frac{11}{24}>\frac{10}{24}$

$\Rightarrow P\left(E^C\cap F^C\cap G^C\right)>\frac{5}{12}$

$\Rightarrow$ (D) is incorrect

14. For any 3 × 3 matrix M, let |M| denote the determinant of M. Let I be the 3 × 3 identity matrix. Let E and F be two 3 × 3 matrices such that (I – EF) is invertible. If $G = (I - EF)^{-1}$, then which of the following statements is (are) TRUE ?

(A) |FE| + |I – FE| |FGE|
(B) (I – FE)(I + FGE) = 1
(C) EFG = GEF
(D) (I – FE)(I – FGE) = 1

Ans. (A,B,C)

Sol. $G = (I - EF)^{-1}$

$\Rightarrow I-EF = G^{-1}$

Now, $G.G^{-1} = G^{-1}.G$

G(I–EF) = (I–EF).G

G – GEF = G – EFG

GEF = EFG ....(1)

(C) is correct

Now,

(A) (I–FE) (FGE)

= FGE – FEFGE ....(2)

$I - EF = G^{-1}$

$\Rightarrow G(I-EF) = G.G^{-1}$

$\Rightarrow$ G – GEF = I

$\Rightarrow$ G – EFG = I [From (1)]

$\Rightarrow$ G – I = EFG ....(3)

Put G–I = EFG in (2), we get

FGE – FEFGE
= FGE – F(G–I)E
= FGE – FGE +FE
= FE
⇒ (I–FE) (FGE) = FE
⇒ |I – FE| |FGE| = |FE|
⇒ (A) is correct
(B) (I–FE) (I + FGE)
= I + FGE – FE – FE FGE
= I + FGE – FE – F (G –I)E (from (3))
= I + FGE – FE – FGE + FE
= I
⇒ (B) is correct
(C) (I–FE) (I – FGE)
= I – FGE – FE – FE FGE
= I – FGE – FE + F(G –I)E (from (3))
= I – FGE – FE + FGE – FE
= I – 2FE
⇒ (D) is incorrect

15. For any positivbe integer n, let Sn :(0, ∞) → R be defined by

$S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$, where for any $x \in R, \cot^{-1}(x) \in (0, \pi)$ and

$\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ Then which of the following statements is (are) TRUE ?

(A) $S_{10}(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right)$, for all x > 0

(B) $\lim_{x\to\infty} \cot(S_n(x)) = x$, for all x > 0

(C) The equation $S_3(x) = \frac{\pi}{4}$ has a root in (0, ∞)

(D) $\tan(S_n(x)) \le \frac{1}{2}$, for all $n \ge 1$ and x > 0

Ans. (A,B)

Sol. $S_n(x) = \sum_{k=1}^{n} \cot^{-1}\left(\frac{1+k(k+1)x^2}{x}\right)$

$$= \sum_{k=1}^{n} \tan^{-1}\left(\frac{(k+1)x - kx}{1+(k+1)xkx}\right)$$

$$= \sum_{k=1}^{n} \tan^{-1}(k+1)x - \tan^{-1} kx$$

$$= (\tan^{-1} 2x - \tan^{-1} x) + (\tan^{-1} 3x - \tan^{-1} 2x) + ... + (\tan^{-1}(n+1)x - \tan^{-1} nx)$$

$S_n(x) = \tan^{-1}(n+1)x - \tan^{-1} x$

(A) $S_{10}(x) = \tan^{-1} 11x - \tan^{-1} x, x > 0$

$$= \tan^{-1}\left(\frac{11x - x}{1+11x.x}\right)$$

$$= \tan^{-1}\left(\frac{10x}{1+11x^2}\right)$$

$$= \cot^{-1}\left(\frac{1+11x^2}{10x}\right)$$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{1+11x^2}{10x}\right) \Rightarrow$ (A) is correct

(B) $\lim_{n\to\infty} \cot(S_n(x)) = \lim_{n\to\infty} \cot(\tan^{-1}(n+1)x - \tan^{-1} x), x > 0$

$$= \lim_{n\to\infty} \cot\left(\tan^{-1} \frac{(n+1)x - x}{1+(n+1)x.x}\right)$$

$$= \lim_{n\to\infty} \cot\left(\tan^{-1} \frac{nx}{1+(n+1)x^2}\right)$$

$$= \lim_{n\to\infty} \cot\left[\tan^{-1} \frac{x}{\frac{1}{n} + \left(1+\frac{1}{n}\right)x^2}\right] = \cot \tan^{-1}\left(\frac{x}{x^2}\right)$$

$$= \cot(\cot^{-1} x) = x$$

$\Rightarrow$ (B) is correct

(C) $S_3(x) = \frac{\pi}{4}$ has a root in $(0, \infty)$

$\tan^{-1} 4x - \tan^{-1} x = \frac{\pi}{4}$

$\Rightarrow \frac{3x}{1+4x^2} = 1 \Rightarrow 4x^2 + 1 = 3x \Rightarrow 4x^2 - 3x + 1 = 0$

$D < 0$

$\Rightarrow$ no real

$\Rightarrow$ (C) option is incorrect

(D) $S_n(x) = \tan^{-1}(n+1)x - \tan^{-1} x$

$= \tan^{-1}\left(\frac{nx + x - x}{1+(n+1)x^2}\right)$

$= \tan^{-1}\left(\frac{nx}{1+(n+1)x^2}\right)$

$\tan(S_n(x)) = \frac{nx}{1+(n+1)x^2} = \frac{n}{\frac{1}{x}+(n+1)x}$

$\Rightarrow$ A.M. $\geq$ G.M.

$\frac{\frac{1}{x}+(n+1)x}{2} \geq \sqrt{\frac{1}{x}.(n+1)x}$

$\frac{1}{x}+(n+1)x \geq 2\sqrt{n+1}$

$\Rightarrow \tan(S_n(x)) = \frac{nx}{1+(n+1)x^2} \leq \frac{n}{2\sqrt{n+1}}$

for $n = 2$

$\frac{n}{2\sqrt{n+1}} = \frac{1}{\sqrt{3}} > \frac{1}{2}$

D is incorrect

16. For any complex number $w = c + id$, let $\arg(w) \in (-\pi, \pi]$ where $i = \sqrt{-1}$. Let α and β be real numbers such that for all complex numbers $z = x + iy$ satisfying $\arg\left(\frac{z+\alpha}{z+\beta}\right) = \frac{\pi}{4}$, the ordered pair (x, y) lies on the circle

$x^2 + y^2 + 5x - 3y + 4 = 0$

Then which of the following statements is (are) TRUE ?

(A) $\alpha = -1$ (B) $\alpha\beta = 4$ (C) $\alpha\beta = -4$ (D) $\beta = 4$

Ans. (B,D)

Sol. $\arg\left(\frac{z+\alpha}{z+\beta}\right)=\frac{\pi}{4}$; α, β are real numbers

$\Rightarrow \arg(z+\alpha)-\arg(z+\beta)=\frac{\pi}{4}$

$\Rightarrow \arg(x+\alpha+iy)-\arg(x+\beta+iy)=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{y}{\alpha+x}\right)-\tan^{-1}\left(\frac{y}{\beta+x}\right)=\frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left[\frac{\frac{y}{\alpha+x}-\frac{y}{\beta+x}}{1+\left(\frac{y}{\alpha+x}\right)\left(\frac{y}{\beta+x}\right)}\right]=\frac{\pi}{4}$

$\Rightarrow \frac{\beta y+xy-\alpha y-xy}{\alpha\beta+x(\alpha+\beta)+x^2+y^2}=\tan\frac{\pi}{4}=1$

$\beta y-\alpha y=\alpha\beta+x(\alpha+\beta)+x^2+y^2$

$\Rightarrow x^2+y^2+x(\alpha+\beta)+y(\alpha-\beta)+\alpha\beta=0$

$x^2+y^2+5x-3y+4=0$

$\Rightarrow \alpha+\beta=5 \qquad \alpha\beta=4$

$\alpha-\beta=-3$

$\alpha=1$

$\beta=4$

(B), (D) are correct

17. For $x\in R$, the number of real roots of the equation $3x^2-4|x^2-1|+x-1=0$ is ___ .

Ans. (4)

Sol. $3x^2-4|x^2-1|+x-1=0$

Case I: $x^2-1\geq 0$

$\Rightarrow x\in(-\infty,-1]\cup[1,\infty)$ ....(1)

$3x^2-4(x^2-1)+x-1=0$

$3x^2-4x^2+4+x-1=0$

$x^2-x-3=0$

$\Rightarrow x=\frac{1\pm\sqrt{13}}{2}$ (Both are acceptable)

Case II: $x^2-1\leq 0$

$x\in[-1,1]$ ....(2)

$3x^2+4(x^2-1)+x-1=0$

$7x^2+x-5=0$

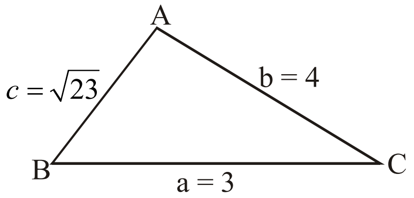
$\Rightarrow x = \dfrac{-1 \pm \sqrt{141}}{14}$ (Both belongs to (2))

Total solution = 4

18. In a triangle ABC, let $AB = \sqrt{23}, BC = 3$ and CA = 4. Then teh value of $\dfrac{\cot A + \cot C}{\cot B}$ is __.

Ans. (2)

Sol.



According to sin rule & cosine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{1}{\lambda} \text{ (say)}$$

$\Rightarrow \sin A = a\lambda$

$\Rightarrow \sin B = b\lambda$

$\Rightarrow \sin C = c\lambda$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \cos C = \frac{a^2 + c^2 - b^2}{2ac}$$

Now, $\dfrac{\cot A + \cot C}{\cot B}$

$$= \frac{\dfrac{\cos A}{\text{Sin } A} + \dfrac{\cos C}{\sin C}}{\dfrac{\cos B}{\sin B}} = \frac{\dfrac{b^2 + c^2 - a^2}{2bc(a\lambda)} + \dfrac{a^2 + b^2 - c^2}{2ab(c\lambda)}}{\dfrac{a^2 + c^2 - b^2}{2ac(b\lambda)}}$$

$$= \frac{b^2 + c^2 - a^2 + a^2 + b^2 - c^2}{a^2 + c^2 - b^2} = \frac{2b^2}{a^2 + c^2 - b^2}$$

$$= \frac{2 \times 16}{9 + 23 - 16} = \frac{32}{16} = 2$$

19. Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in three-dimensional space, where $\vec{u}$ and $\vec{v}$ are unit vectors which are not perpendicular to each other and

$\vec{u}.\vec{w} = 1, \vec{v}.\vec{w} = 1 \; \vec{w}.\vec{w} = 4$

If the volume of the parallelopiped, whose adjacent sides are represented by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$, is $\sqrt{2}$, then the value of $|3\vec{u} + 5\vec{v}|$ is ____ .

Ans. (7)

Sol. $\vec{u}\cdot\vec{v}\neq 0$

$\vec{u}\cdot\vec{w}=1,\ \vec{v}\cdot\vec{w}=1,\ \vec{w}\cdot\vec{w}=4,\ [\vec{u}\vec{v}\vec{w}]=\sqrt{2}$

$|3\vec{u}+5\vec{v}|=?,\ \vec{u}\ \&\ \vec{v}$ are unit vectors

$[\vec{u}\vec{v}\vec{w}]=\sqrt{2}$

$\Rightarrow [\vec{u}\vec{v}\vec{w}]^2=2$

$$\Rightarrow \begin{vmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v} & \vec{u}\cdot\vec{w} \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{v} & \vec{v}\cdot\vec{w} \\ \vec{w}\cdot\vec{u} & \vec{w}\cdot\vec{v} & \vec{w}\cdot\vec{w} \end{vmatrix}=2$$

$$\Rightarrow \begin{vmatrix} 1 & x & 1 \\ x & 1 & 1 \\ 1 & 1 & 4 \end{vmatrix}=2 \quad (\text{Let } \vec{u}\cdot\vec{v}=x)$$

$\Rightarrow 1(4-1)-x(4x-1)+1(x-1)=2$

$\Rightarrow 3-4x^2+x+x-1=2$

$\Rightarrow 4x^2-2x=0$

$\Rightarrow x=\left\{0,\frac{1}{2}\right\}\Rightarrow \vec{u}\cdot\vec{v}=0\,(\textit{rejected}) \quad \textit{or} \quad \vec{u}\cdot\vec{v}=\frac{1}{2}$

Now, $|3\vec{u}+5\vec{v}|^2=9\vec{u}\cdot\vec{u}+25\vec{v}\cdot\vec{v}+30\vec{u}\cdot\vec{v}$

$9+25+30\times\frac{1}{2}$

$=49$

$\Rightarrow |3\vec{u}+5\vec{v}|=7$