JEE Advanced 2020

Question Paper With Text Solutions

PAPER-2

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE ADVANCED SEP 2020 | 27 SEP PAPER-2

MATHS

SECTION-1 (Maximum Marks : 18)

* This section contains SIX (06) questions.
* The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, BOTH INCLUSIVE.
* For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
* Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +3 If ONLY the correct integer is entered.

Zero Marks : 0 If the question is unanswered.

Negative marks : –1 In all other cases.

1. For a complex number z, let Re(z) denote the real part of z. Let S be the set of all complex numbers z satisfying $z^4 - |z|^4 = 4\,i\,z^2$, where $i = \sqrt{-1}$. Then the minimum possible value of $|z_1 - z_2|^2$, where $z_1, z_2 \in S$ with $Re(z_1) > 0$ and $Re(z_2) < 0$, is _____

Ans. 8

Sol. $z^4 - |z|^4 = 4i\,z^2$

$z^4 - z^2\bar{z}^2 = 4i\,z^2$ $\qquad\qquad z^2 \neq 0$

$z^2 - \bar{z}^2 = 4i$

$(z+\bar{z})(z-\bar{z}) = 4i$ $\qquad$ (1)

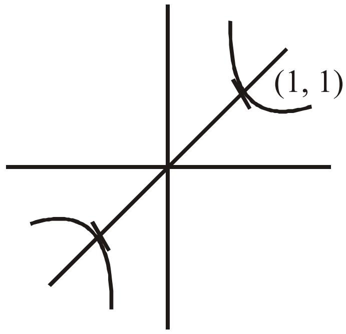
$z = x + iy$

then by (1)

$4ixy = 4i$

$xy = 1$

It is a hyberbola, z_1 and z_2 are two points on this hyberbola



$|z_1 - z_2|_{min}$ = T.A. of hyberbola = $2\sqrt{2}$

$|z_1 - z_2|^2_{min} = 8$

2. The probability that a missile hits a target successfully is 0.75. In order to destroy the target completely, at least three successful hits are required. Then the minimum number of missiles that have to be fired so that the probability of completely destroying the target is NOT less than 0.95, is ————

Ans. 6

Sol. By probability distribution

X = no of hits

$p = \frac{3}{4}$ (probability of success)

$p(X \geq 3) \geq 0.95$

$1 - \left(p(X=0) + p(X=1) + p(X=2)\right) \geq .95$

$$\frac{5}{100} \geq {}^nc_0\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^n + {}^nc_1\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^{n-1} + {}^nc_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^{n-2}$$

$$\frac{5}{100} \geq \frac{1}{4^n} + \frac{3n}{4^n} + \frac{n(n-1)}{2} \cdot \frac{9}{4^n}$$

$$\frac{4^n}{20} \geq \frac{2 + 6n + 9n^2 - 9n}{1}$$

$$\frac{2^{2n-1}}{5} \ge 9n^2 - 3n + 2$$

$$2^{2n-1} \ge 5\left(9n^2 - 3n + 2\right)$$

$n = 3 \qquad 32 \ge 5 \times 74$

$n = 4 \qquad 128 \ge 5 \times 132$

$n = 5 \qquad 512 \ge 5 \times 212$

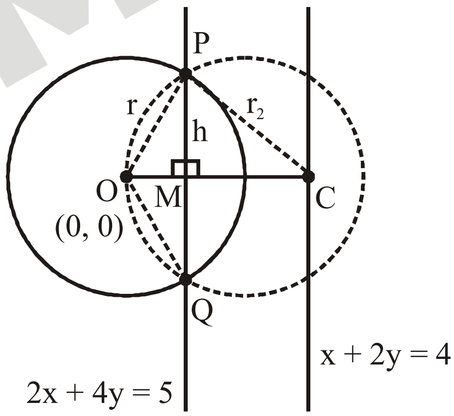
$n = 6 \qquad 2048 \ge 5 \times 308$

hence minimum hit required are '6'.

3. Let O be the centre of the circle $x^2 + y^2 = r^2$, where $r > \frac{\sqrt{5}}{2}$. Suppose PQ is a chord of this circle and the equation of the line passing through P and Q is $2x + 4y = 5$. If the centre of the circumcircle of the triangle OPQ lies on the line $x + 2y = 4$, then the value of r is _____

Ans. 2

Sol.



MC = distance between two lines

$$MC = \frac{3}{2\sqrt{5}}$$

$$OC = r_2 = \frac{4}{\sqrt{5}}$$

$h^2 + OM^2 = r^2 \qquad$ -(1)

$h^2 + CM^2 = r_2^2$ -(2)

$h^2 + \frac{25}{20} = r^2$ -(1)

$h^2 + \frac{9}{20} = \frac{16}{5}$ -(2)

by (1) - (2)

$r^2 = 4$

$r = 2$

4. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2 × 2 matrix such that the trace of A is 3 and the trace of A^3 is –18, then the value of the determinant of A is_____

Ans. 5

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $a + d = 3$

$d = 3 - a$

$A = \begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}$

$|A| = a(3-a) - bc$

$|A| = 3a - a^2 - bc$

$A^2 = \begin{bmatrix} a & b \\ c & 3-a \end{bmatrix} \begin{bmatrix} a & b \\ c & 3-a \end{bmatrix} = \begin{bmatrix} a^2+bc & 3b \\ 3c & (3-a)^2+bc \end{bmatrix}$

$A^3 = \begin{bmatrix} a^2+bc & 3b \\ 3c & (3-a)^2+bc \end{bmatrix} \begin{bmatrix} a & b \\ c & 3-a \end{bmatrix}$

$\text{tr}(A^3) = a^3 + abc + 3bc + 3bc + (3 - a)[(3 - a)^2 + bc] = -18$

$\text{tr}(A^3) = a^3 + abc + 6bc + (3 - a)^3 + 3bc - abc = -18$

$\text{tr}(A^3) = a^3 + 9bc + (3 - a)^3 = -18$

$\text{tr}(A^3) = a^3 + 9bc + 27 - a^3 - 9a(3 - a) = -18$

$9bc + 27 - 27a + 9a^2 = -18$

$bc + 3 - 3a + a^2 = -2$

$5 = 3a - a^2 - bc$

hence

$|A| = 3a - a^2 - bc$

$|A| = 5$

5. Let the functions $f : (-1, 1) \to R$ and $g : (-1, 1) \to (-1, 1)$ be defined by

$f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$,

where [x] denotes the greatest integer less than or equal to x. Let $f \circ g : (-1, 1) \to R$ be the composite function defined by $(f \circ g)(x) = f(g(x))$. Suppose c is the number of points in the interval (–1, 1) at which $f \circ g$ is NOT continuous, and suppose d is the number of points in the interval (–1, 1) at which $f \circ g$ is NOT differentiable. Then the value of c + d is ———

Ans. 4

Sol. $f(x) = |2x - 1| + |2x + 1|$

$g(x) = x - [x]$

$g(x) = [x] + \{x\} - [x]$

$g(x) = \{x\}$

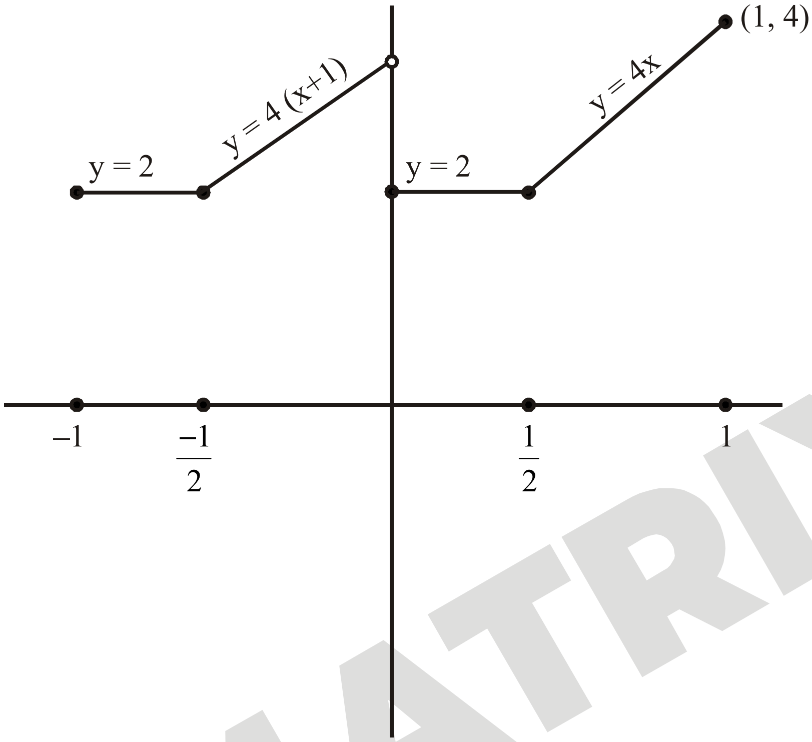
$$f(x) = \begin{cases} 4x & : x \geq \frac{1}{2} \\ 2 & : \frac{-1}{2} < x < \frac{1}{2} \\ -4x & : x \leq \frac{-1}{2} \end{cases}$$

$$f\left(g(x)\right)=\begin{cases} 4\{x\} & :\{x\}\ge \dfrac{1}{2} \\ 2 & :\dfrac{-1}{2}<\{x\}<\dfrac{1}{2} \\ -4\{x\} & :\{x\}\le \dfrac{-1}{2} \end{cases}$$

$$f\left(g(x)\right)=\begin{cases} 4\{x\} & :\{x\}\ge \dfrac{1}{2} \\ 2 & :0\le \{x\}<\dfrac{1}{2} \end{cases}$$

$$f\left(g(x)\right)=\begin{cases} 4\{x\} & :x\in\left[\dfrac{1}{2},1\right)\cup\left[\dfrac{-1}{2},0\right) \\ 2 & :\left[0,\dfrac{1}{2}\right)\cup\left(\dfrac{-1}{2},-1\right) \end{cases}$$

$$f\left(g(x)\right)=\begin{cases} 4x & :x\in\left[\dfrac{1}{2},1\right) \\ 4\left(x+1\right) & :x\in\left[\dfrac{-1}{2},0\right) \\ 2 & :x\in\left[0,\dfrac{1}{2}\right)\cup\left(\dfrac{-1}{2},-1\right) \end{cases}$$



Not continuous at x = 0

Not differetiable at x = $\pm\frac{1}{2}, 0$

c = 1

d = 3

c + d = 1 + 3 = 4

6. The value of the limit

$$\lim_{x \to \frac{\pi}{2}} \frac{4\sqrt{2}\left(\sin 3x + \sin x\right)}{\left(2\sin 2x \sin\frac{3x}{2} + \cos\frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2}\cos 2x + \cos\frac{3x}{2}\right)}$$

Ans. 8

Sol. $$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2\sin 2x \sin\frac{3x}{2} + \cos\frac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2}\cos 2x + \cos\frac{3x}{2}\right)}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}(2\sin 2x \cos x)}{\left[\cos\frac{x}{2} - \cos\frac{7x}{2} + \cos\frac{5x}{2}\right] - \left[\sqrt{2}\left(2\cos^2 x\right) + \cos\frac{3x}{2}\right]}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{4\sqrt{2}(2\sin 2x \cos x)}{\cos\frac{x}{2} - \cos\frac{3x}{2} + \cos\frac{5x}{2} - \cos\frac{7x}{2} - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}\sin x \cos^2 x}{2\sin x \sin\frac{x}{2} + 2\sin 3x \sin\frac{x}{2} - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}\sin x \cos^2 x}{2\sin\frac{x}{2}(\sin 3x + \sin x) - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}\sin x \cos^2 x}{2\sin\frac{x}{2}(2\sin 2x \cos x) - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}\sin x \cos^2 x}{8\sin\frac{x}{2}\sin x \cos^2 x - 2\sqrt{2}\cos^2 x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{16\sqrt{2}}{8\times\frac{1}{\sqrt{2}} - 2\sqrt{2}} = 8$$

SECTION-2 (Maximum Marks : 24)

* This section contains **SIX** (06) questions.
* Each question has **FOUR** options. ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
* For each question, choose the option(s) corresponding to (all) the correct answer(s).

* Answer to each question will be evaluated according to the following marking scheme.

Full Marks : +4 If only (all) the correct option(s) is (are) chosen.

Partial Marks : +3 If all the four options are correct but ONLY three options are chosen.

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen and both of which are correct.

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option.

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : –2 In all other cases.

is :

7. Let b be a nonzero real number. Suppose $f : R \rightarrow R$ is a differentiable function such that $f(0) = 1$.

If the derivative f' of f satisfies the equation

$$f'(x) = \frac{f(x)}{b^2 + x^2}$$

for all $x \in R$, then which of the following statements is/are TRUE ?

(A) If $b > 0$, then f is an increasing function

(B) If $b < 0$, then f is an decreasing function

(C) $f(x)\,f(-x) = 1$ for all $x \in R$

(D) $f(x) - f(-x) = 0$ for all $x \in R$

Sol. f is non zero

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{x^2 + b^2}\,dx$$

$$ln(f(x)) = \frac{1}{b}\tan^{-1}\frac{x}{b} + c$$

$$0 = 0 + c \quad \{\because f(0) = 1\}$$

$$f(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}$$

$f(-x) = e^{\frac{-1}{b}\tan^{-1}\frac{x}{b}}$

(A) f(x) is positive function for all $x \in R$

$f'(x) = \dfrac{f(x)}{x^2 + b^2}$

$f'(x)$ is also positive function
so f(x) is increasing function

(d) $f(x) - f(-x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}} - e^{\frac{-1}{b}\tan^{-1}\frac{x}{b}} \neq 0$

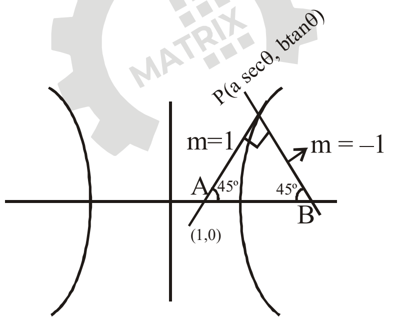
for all $x \in R$
(c) $f(x)\,f(-x) = 1$
$e^0 = 1$

$1 = 1$

8. Let a and b be positive real numbers such that a > 1 and b < a. Let P be a point in the first quadrant that lies on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Suppose the tangent to the hyperbola at P passes through the point (1, 0), and suppose the normal to the hyperbola at P cuts off equal intercepts on the coordinate axes. Let Δ denote the area of the triangle formed by the tangent at P, the normal at P and the x-axis. If e denotes the eccentricity of the hyperbola, then which of the following statements is/are TRUE?

(A) $1 < e < \sqrt{2}$ (B) $\sqrt{2} < e < 2$ (C) $\Delta = a^4$ (D) $\Delta = b^4$

Sol.



equation of tangent ⇒ T = 0

$\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$

passing through (1, 0)

$a = \sec\theta$

$m = \frac{b\sec\theta}{a\tan\theta} = 1$

$b = \tan\theta$

$p(a^2, b^2)$ or $p(\sec^2\theta, \tan^2\theta)$

$b^2 = a^2(e^2 - 1)$

$\tan^2\theta = \sec^2\theta\,(e^2 - 1)$

$e^2 - 1 = \sin^2\theta$

$0 < \sin^2\theta < 1$

$1 < 1 + \sin^2\theta < 2$

$1 < e^2 < \sqrt{2}$

$1 < e < \sqrt{2}$

Area of $\Delta PAB = \frac{1}{2}(AP)(PB)$

$= \frac{1}{2}(AP)^2 \qquad \{\because PA = PB\}$

$= \frac{1}{2}\left\{(\sec^2\theta - 1)^2 + (\tan^2\theta - 0)^2\right\}$

$= \tan^4\theta = b^4$

9. Let $f: R \to R$ and $g: R \to R$ be functions satisfying

$f(x+y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$

for all $x, y \in R$. If $\lim_{x\to 0} g(x) = 1$, then which of the following statements is/are TRUE ?

(A) f is differentiable at every $x \in R$

(B) If $g(0) = 1$, then g is differentiable at every $x \in R$

(C) The derivative $f'(1)$ is equal to 1

(D) The derivative $f'(0)$ is equal to 1

Sol. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x+h)-f(x+0)}{h}$

$= \lim_{h\to 0}\frac{f(x)+f(h)+f(x)f(h)-f(x)-f(0)-f(x)f(0)}{h}$

$f'(x) = \lim_{h\to 0}\frac{f(h)-f(0)+f(x)\left(f(h)-f(0)\right)}{h}$

$f'(x) = \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right)\left(f(x)+1\right)$

$f'(x) = \lim_{h\to 0}\left(\frac{hg(h)-(0)g(0)}{h}\right)\left(f(x)+1\right)$

$f'(x) = \lim_{h\to 0} g(h).(f(x)+1)$

$f'(x) = f(x) + 1$

$\int\frac{f'(x)}{f(x)+1}dx = \int 1dx$

$\ln(f(x)+1) = x + c$ { $\therefore$ $f(0) = 0$}

$c = 0$

$f(x) = e^x - 1$

f is differentiable at every $x \in R$

$g(x) = \frac{f(x)}{x} = \frac{e^x-1}{x}$

Check at x = 0

$g'(0) = \lim_{h-0}\frac{g(0+h)-g(0)}{h}$

$\lim_{h-0}\frac{\frac{e^h-1}{h}-1}{h}$

$$\lim_{h\to 0}\frac{1+h+\frac{h^2}{2!}+...............-1-h}{h}$$

$g'(0) = \frac{1}{2}$

If g (0) = 1 then g is differentiable at every $x \in R$

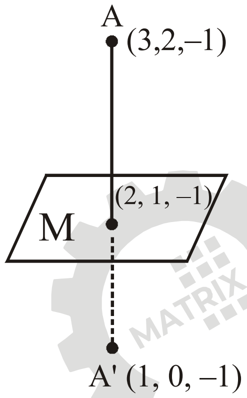
$f'(1) = f(1) + 1 = e - 1 + 1 = e$

$f'(0) = f(0) + 1 = 1$

10. Let α , β , γ , δ be real numbers such that $\alpha^2+\beta^2+\gamma^2 \neq 0$ and $\alpha + \gamma = 1$. Suppose the point (3,2, –1) is the mirror image of the point (1,0, –1) with respect to the plane $\alpha x + \beta y + \gamma z = \delta$. Then which of the following statements is/are TRUE?

(A) $\alpha + \beta = 2$ (B) $\delta - \gamma = 3$ (C) $\delta + \beta = 4$ (D) $\alpha + \beta + \gamma = \delta$

Sol.



Normal vector

$\vec{n} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$\vec{n} = \overrightarrow{A'A}$

$\vec{n} = 2\hat{i} + 2\hat{j} + 0\hat{k}$

$\frac{\alpha}{2} = \frac{\beta}{2} = \frac{\gamma}{0} = \lambda$

$\alpha + \gamma = 1$

$\alpha = 1 = \beta \quad \alpha = \beta$ and $\gamma = 0$

$\alpha x + \beta y + \gamma z = \delta$

m(2, 1, –1)

$2\beta + \beta + 0 = \delta$

$\delta = 3$

Ans A, B, C.

11. Let a and b be positive real numbers. Suppose $\overrightarrow{PQ} = a\hat{i} + b\hat{j}$ and $\overrightarrow{PS} = a\hat{i} - b\hat{j}$ are adjacent sides of a parallelogram PQRS. Let $\vec{u}$ and $\vec{v}$ be the projection vectors of $\vec{w} = \hat{i} + \hat{j}$ along $\overrightarrow{PQ}$ and $\overrightarrow{PS}$, respectively. If $|\vec{u}| + |\vec{v}| = |\vec{w}|$ and if the area of the parallelogram PQRS is 8, then which of the following statements is/are TRUE?

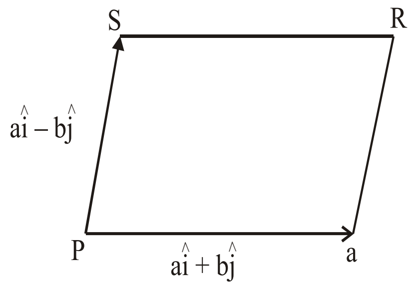
(A) a + b = 4

(B) a – b = 2

(C) The length of the diagonal PR of the parallelogram PQRS is 4

(D) $\vec{w}$ is an angle bisector of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

Sol.



$$\vec{u} = \left(\frac{\overrightarrow{w}.\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\right)\hat{P}S = \left(\frac{a-b}{\sqrt{a^2+b^2}}\right)\left(\frac{a\hat{j}-b\hat{j}}{\sqrt{a^2+b^2}}\right)$$

$$\vec{v} = \left(\frac{\overrightarrow{w}.\overrightarrow{PS}}{|\overrightarrow{PS}|}\right)\hat{P}S = \left(\frac{a-b}{\sqrt{a^2+b^2}}\right)\left(\frac{a\hat{j}-b\hat{j}}{\sqrt{a^2+b^2}}\right)$$

$$|\vec{u}| + |\vec{v}| = |\overrightarrow{w}|$$

$$\frac{a+b}{\sqrt{a^2+b^2}} + \frac{|a-b|}{\sqrt{a^2+b^2}} = \sqrt{2}$$

Case - I $a > b$

$$2a = \sqrt{2}\sqrt{a^2+b^2}$$
$$2a^2 = a^2 + b^2$$
$$a^2 = b^2$$
$$a = b$$

Case - II $a < b$

$$2a = \sqrt{2}\sqrt{a^2+b^2}$$
$$a = b$$

$$\text{Area of the parallelogram} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & -b & 0 \\ a & b & 0 \end{vmatrix} = 8$$

$$\left|\hat{k}.2ab\right| = 8$$

$ab = 4$

and $a = b$

$a = b = 2$

(A) $a+b = 4$

(C) $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = 2a\hat{i}$

$$\left|\overrightarrow{PR}\right| = 4$$

(D) $\overrightarrow{PQ} + \overrightarrow{PS} = 2a\hat{i} + \lambda\vec{\omega}$

$$\overrightarrow{PQ} - \overrightarrow{PS} = 2b\hat{j} \neq \lambda\vec{\omega}$$

12. For nonnegative integers s and r, let

$$\binom{s}{r} = \begin{cases} \dfrac{s!}{r!(s-r)!} & \text{if } r \le s, \\ 0 & \text{if } r > s. \end{cases}$$

For positive integers m and n, let

$$g(m,n) = \sum_{p=0}^{m+n} \frac{f(m,n,p)}{\binom{n+p}{p}}$$

where for any nonnegative integers p,

$$f(m, n, p) = \sum_{i=0}^{p} \binom{m}{i}\binom{n+i}{p}\binom{p+n}{p-i}$$

Then which of the following statements is / are TRUE ?

(A) g (m, n) = g (n, m) for all positive integers m, n

(B) g (m, n + 1) = g (m + 1, n) for all positive integers m, n

(C) g (2m, 2n) = 2g (m, n) for all positive integers m, n

(D) g (2m, 2n) = $(g(m, n))^2$ for all positive integers m, n

Sol. $f(m,n,p) = \sum_{i=0}^{p}\left({}^mC_i \frac{(n+i)!}{(n+i-p)!p!} \times \frac{(p+n)!}{(n+i)!(p-i)!}\right)$

$$= \sum_{i=0}^{p} {}^mC_i \left(\frac{n!}{(n+i-p)!(p-i)!}\right)\left(\frac{p+n!}{(n)!(p)!}\right)$$

$$= {}^{n+p}C_p \sum_{i=0}^{p} {}^mC_i\, {}^nC_{p-i}$$

$$f(m,n,p) = {}^{n+p}C_p \left({}^mC_0\, {}^nC_p + {}^mC_1\, {}^nC_{p-1} + \ldots\ldots + {}^mC_p\, {}^nC_0\right)$$

$$= {}^{n+p}C_p \cdot {}^{m+n}C_p$$

$$g(m,n) = \sum_{p=0}^{m+n} \frac{{}^{n+p}C_p \;\; {}^{m+n}C_p}{{}^{n+p}C_p} = 2^{m+n}$$

SECTION-3 (Maximum Marks : 24)

* This section contains SIX (06) questions. The answer to each question is a NUMERICAL VALUE.
* For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places truncate/round-off the value to TWO decimal placed.
* Answer to each question will be evaluated according to the following marking scheme :

  Full Marks : +4 If ONLY the correct numerical value is entered.

  Zero Marks : 0 In all other cases.

13. An engineer is required to visit a factory for exactly four days during the first 15 days of every month and it is mandatory that **no** two visits take place on consecutive days. Then the number of all possible ways in which such visits to the factory can be made by the engineer during 1-15 June 2021 is _____

Sol. Out of 15 days, we have to choose four days such that no two are consecutive.

Let us reduce 4 days out of 15, so Remaining days = 15 – 4 = 11.

Let the remaining days are d_1, d_2,...... d_{11}

$\downarrow d_1 \downarrow d_2 \downarrow d_3 \downarrow ...... \downarrow d_{11} \downarrow$

Among the 12 gaps in these 11 days, we have to pick any 4, so

No. of ways $^{12}c_4 = \frac{12\times11\times10\times9}{4\times3\times2} = 495$

14. In a hotel, four rooms are available. Six persons are to be accommodated in these four rooms in such a way that each of these rooms contains at least one person and at most two persons. Then the number of all possible ways in which this can be done is _____.

Sol. The only ways in distributing 6 - persons in 4 rooms, such that each room contains at least 1 and at most 2 person is (1, 1, 2, 2).

No. of ways of grouping $= \frac{6!}{1!1!(2!).2!2!(2!)}$

$= 45$

No. of ways of arranging these 4 groups in 4 rooms = 4! = 24

So total no. of ways = 45 × 24 = 1080

15. Two fair dice, each with faces numbered 1,2,3,4,5 and 6, are rolled together and the sum of the numbers on the faces is observed. This process is repeated till the sum is either a prime number or a perfect square. Suppose the sum turns out to be a perfect square before it turns out to be a prime number. If p is the probability that this perfect square is an odd number, then the value of 14p is _____

Sol. E_1 : Sum on pair of dice is Prime

E_2 : Sum on pair of dice is perfect square

E_1 : Sum = 2 {(1, 1)}

= 3 {(1, 2), (2, 1)}

= 5 {(1, 4), (2, 3), (3, 2), (4, 1)}

= 7 {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

= 11 {(5, 6), (6, 5)}

$n(E_1) = 1 + 2 + 4 + 6 + 2 = 15$

$$P(E_1) = \frac{15}{36}$$

E_2 : sum = 4 { (1, 3), (2, 2), (3, 1)}

= 9 {(3, 6), (4, 5), (5, 4), (6, 3)}

$n(E_2) = 3 + 4 = 7$,

$$P(E_2) = \frac{7}{36}$$

$$P(E_1 \cup E_2) = \frac{22}{36}$$

$$P\left(\overline{E_1 \cup E_2}\right) = \frac{14}{36}$$

The desired conditional Probability

$$p = P\left(\frac{\text{The perfect square is odd}}{\text{The Sum is a perfect square before prime}}\right)$$

$$= \frac{\text{P(odd Perfect Square before prime)}}{\text{P(Perfect square before prime)}} = \frac{P_1}{P_2}$$

Since odd perfect square is 9

$$P_1 = \frac{4}{36} + \frac{14}{36} \times \frac{4}{36} + \left(\frac{14}{36}\right) \times \frac{14}{36} \times \frac{4}{36} + \ldots$$

$$= \frac{\frac{4}{36}}{1 - \frac{14}{36}} = \frac{4}{22} \ldots.(1)$$

$$P_2 = \frac{7}{36} + \frac{14}{36} \times \frac{7}{36} + \left(\frac{14}{36}\right) \times \frac{14}{36} \times \frac{7}{36} + \ldots.$$

$$= \frac{\frac{7}{36}}{1 - \frac{14}{36}} = \frac{7}{22} \ldots.(2)$$

$$\Rightarrow p = \frac{P_1}{P_2} = \frac{4}{7}.$$

$$\Rightarrow 14p = \frac{4}{7} \times 14 = 8 \cdot$$

16. Let the function $f : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \frac{4^x}{4^x + 2}.$$

Then the value of

$$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \ldots..+ f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$$

is.

Sol. $f(x)=\frac{4^x}{4^x+2}$; Replace $x\to 1-x$

$$f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}=\frac{4/4^x}{4/4^x+2}=\frac{2}{4^x+2}$$

$$f(x)+f(1-x)=\frac{4^x}{4^x+2}+\frac{2}{4^x+2}=1\ \forall x.$$

So, $f\left(\frac{1}{40}\right)+f\left(\frac{39}{40}\right)=f\left(\frac{2}{40}\right)+f\left(\frac{38}{40}\right)=...=f\left(\frac{19}{40}\right)+f\left(\frac{21}{40}\right)=1$

$$\Rightarrow f\left(\frac{1}{40}\right)+f\left(\frac{2}{40}\right)+....f\left(\frac{19}{40}\right)+f\left(\frac{20}{40}\right)+f\left(\frac{21}{40}\right)+....+f\left(\frac{39}{40}\right)-f\left(\frac{1}{2}\right)=19+f\left(\frac{20}{40}\right)-f\left(\frac{1}{2}\right)$$

$= 19$

17. Lef f : $\mathbb{R}\to\mathbb{R}$ be a differentiable function such that its derivative f' is continuous and $f(\pi)=-6$.

If F: $[0, \pi]\to\mathbb{R}$ is defined by $F(x)=\int_0^x f(t)dt$, and if

$$\int_0^{\pi}(f'(x)+F(x))\cos x\,dx=2,$$

then the value fo $f(0)$ is ____

Sol. $F(x)=\int_0^x f(t)\,dt$

differentiating wrt x

$F'(x) = f(x)$ ......(1)

$$\int_0^{\pi}\left(f'(x)\cos x+F(x)\cos x\right)dx=2$$

$$\Rightarrow\int_0^{\pi}f'(x)\cos x\,dx+\int_0^{\pi}F(x)\cos x\,dx=2$$

Using intgeration by parts

$$\left[f(x)\cos x\right]_0^{\pi} + \int_0^{\pi} \sin x.f(x)\,dx + \left[F(x).\sin x\right]_0^{\pi} - \int_0^{\pi} \sin x\, F'(x)\,dx = 2$$

Using (1)

$$\left[f(x)\cos x\right]_0^{\pi} + \int_0^{\pi} \sin x.f(x)\,dx + \left[F(x).\sin x\right]_0^{\pi} - \int_0^{\pi} \sin x\, f(x)\,dx = 2$$

$$\Rightarrow -f(\pi) - f(0) + F(\pi).0 - F(0).0 = 2$$

$$\Rightarrow -f(\pi) - f(0) = 2$$

$$\Rightarrow 6 - f(0) = 2$$

$$\Rightarrow f(0) = 4 \text{ Ans.}$$

18. Let the function $f : (0, \pi) \to \mathbb{R}$ be defined by

$$f(\theta) = (\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^4.$$

Suppose the function f has a local minimum at θ precisely when $\theta \in \{\lambda_1\pi, \ldots \lambda_\rho\pi\}$ where $0 < \lambda_1 < \ldots .\lambda_r < 1$. Then the value of $\lambda_1 + \ldots + \lambda_r$ is ______.

Sol. $f(\theta) = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta + \left((\sin\theta - \cos\theta)^2\right)^2$

$= 1 + \sin 2\theta + (1 - \sin 2\theta)^2$

$= 1 + \sin 2\theta + 1 - 2\sin 2\theta + \sin^2 2\theta$

$= \sin^2 2\theta - \sin 2\theta + 2$

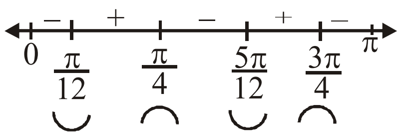
$$\Rightarrow f(\theta) = \left(\sin 2\theta - \frac{1}{2}\right)^2 + \frac{7}{4}.$$

$$\Rightarrow f'(\theta) = 4\cos 2\theta\left(\sin 2\theta - \frac{1}{2}\right)$$

$f'(\theta) = 0$

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$$

sign of $f'(\theta)$:



For local minimum

$\Rightarrow \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$ So $\lambda_1 = \frac{1}{12} \& \lambda_2 = \frac{5}{12}$.

$\lambda_1 + \lambda_2 = \frac{1+5}{12} = \frac{6}{12} = \frac{1}{2} = 0.5.$

2nd Method:

$f(\theta) = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta + \left((\sin\theta - \cos\theta)^2\right)^2$

$= 1 + \sin 2\theta + (1 - \sin 2\theta)^2$

$= 1 + \sin 2\theta + 1 - 2\sin 2\theta + \sin^2 2\theta$

$= \sin^2 2\theta - \sin 2\theta + 2$

$\Rightarrow f(\theta) = \left(\sin 2\theta - \frac{1}{2}\right)^2 + \frac{7}{4}.$

For local minimum

$\sin 2\theta - \frac{1}{2} = 0$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\theta \in (0, \pi) \Rightarrow 2\theta \in (0, 2\pi)$

$\Rightarrow 2\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$ So $\lambda_1 = \frac{1}{12} \& \lambda_2 = \frac{5}{12}$.

$\lambda_1 + \lambda_2 = \frac{1+5}{12} = \frac{6}{12} = \frac{1}{2} = 0.5.$