JEE Advanced 2020

Question Paper With Text Solutions

PAPER-1

MATHEMATICS

MATRIX

JEE Main & Advanced | XI-XII Foundation| VI-X Pre-Foundation

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JEE ADVANCED SEP 2020 | 27 SEP PAPER-1

MATHS

SECTION-1 (Maximum Marks : 18)

* This section contains FOUR (06) questions.
* Each question has FOUR options ONLY ONE of these four options is the correct answer.
* For each question, choose the correct option corresponding to the correct answer.
* Answer to each question will be evaluated according to the following marking scheme :

Full Marks : +3 If ONLY the correct option is chosen.

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : –1 In all other cases.

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$. Then the value of

$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$

is :

(A) 0 (B) 8000 (C) 8080 (D) 16000

Ans. (D)

Sol. $x^2 + 20x - 2020 = 0$ a, b are its roots

$x^2 - 20x + 2020 = 0$ c,d are its roots

$a + b = -20$ $ab = -2020$

$c + d = 20$ $cd = 2020$

$E = ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$

$= a^2c - ac^2 + a^2d - ad^2 + b^2c + b^2d - bd^2$

$= a^2(c + d) + b^2(c + d) - c^2(a + b) - d^2(a + b)$

$= (a^2 + b^2)(c + d) - (a + b)(c^2 + d^2)$

$= ((a + b)^2 - 2ab)(c + d) - (a + b)((c + d)^2 - 2cd)$

$= (400 + 4040)(20) - (-20)(400 - 4040)$

$= 88800 - 72800 = 16000$

2. If the function f : **R** → **R** is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is TRUE?

(A) f is one-one, but NOT onto
(B) f is onto, but NOT one-one
(C) f is BOTH one-one and onto
(D) f is NEITHER one-one NOR onto

Ans. (C)

$$f(x) \begin{cases} x(x - \sin x) & x \geq 0 \\ -x(x - \sin x) & x < 0 \end{cases}$$

$\forall x \geq 0$

$f'(x) = 2x - (\sin x + x\cos x)$
$= (x - \sin x) + x(1 - \cos x) \geq 0$
f (x) is increasing

$f(x))_{min} = f(0) = 0$

$x \to \infty \Rightarrow f(x) \to \infty$

$\Rightarrow \forall x \geq 0 \Rightarrow f(x) \in [0, \infty)$
$\forall x < 0$

$\because$ f is odd $\Rightarrow$ f (x) is also increasing

rangs of (x) for x < 0

$f(x) \in (-\infty, 0)$

Rangs of $f(x) = (-\infty, \infty) \Rightarrow f(x)$ is into

f (x) is increasing $\forall x \in R \Rightarrow$ f is one – one

3. Let the functions f: **R** → **R** and g: **R** → **R** be defined by

$f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$.

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

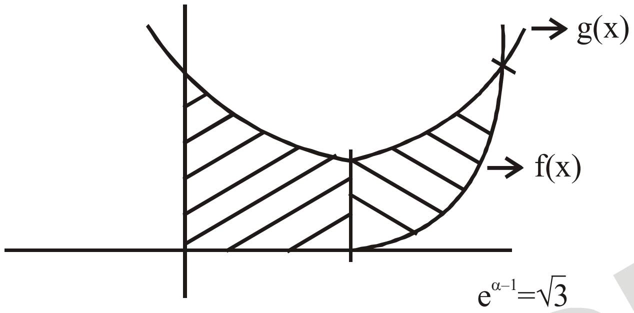
(A) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$
(B) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$
(C) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$
(D) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$

Ans. (A)

Sol. $f(x) \rightarrow e^{x-1} - e^{1-x} \quad x \ge 1$

$\quad\quad \rightarrow 0 \quad\quad x < 1$

$$g(x) = \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)$$



$$P\left(1 + \ln\sqrt{3}, \frac{2}{\sqrt{3}}\right) \equiv P\left(\alpha, \frac{2}{\sqrt{3}}\right)$$

$$\text{Required Area} = \int_0^1 \frac{e^{x-1} + e^{1-x}}{2}\,dx + \int_1^\alpha f(x) - f(x)\,dx$$

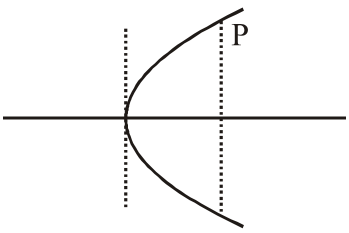
$$= \frac{1}{2}\left(e - \frac{1}{e}\right) + \int_1^\alpha \left(\frac{e^{x-1} + e^{1-x}}{2} - \left(e^{x-1} - e^{1-x}\right)\right)dx$$

$$= \frac{1}{2}\left(e - \frac{1}{e}\right) + \int_1^\alpha \left(\frac{3}{2}e^{1-x} - \frac{e^{x-1}}{2}\right)dx = \left(2 - \sqrt{3}\right) + \frac{1}{2}\left(e - e^{-1}\right)$$

4. Let a, b and λ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

Ans. (A)



p (d, 2λ)
slope of tangent at P
$m_1 = 1$

for ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

$$m)_{(\lambda,2\lambda)}=-1$$

$$-\frac{\lambda b^2}{2\lambda a^2}=-1$$

$$\frac{b^2}{a^2}=2$$

∵ ellipse is vertical

$$e=\sqrt{1-\frac{a^2}{b^2}}=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}$$

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting head in a single toss are 2/3 and 1/3 , respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Ans. (B)

Sol. $x^2 - \alpha x + \beta = 0$

D = 0

$\alpha^2 = 4\beta$

α	β	
0	0	$({}^2C_0\,(2/3)^0\,(1/3)^2)({}^2C_0\,(1/3)^0\,(2/3)^2) = 4/81$
2	1	$({}^2C_2\,(2/3)^2\,(1/3)^0)({}^2C_1\,(1/3)^1\,(2/3)^1) = 16/81$

Required probability = $\frac{4}{81} + \frac{16}{81} = \frac{20}{81}$

6. Consider all rectangles lying in the region

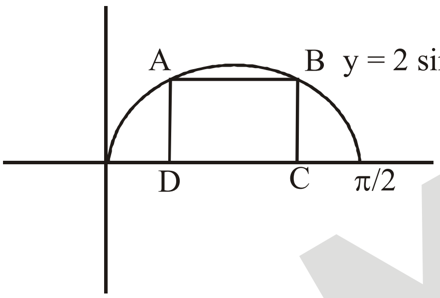
$$\left\{(x, y) \in R \times R : 0 \le x \le \frac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\frac{3\pi}{2}$ (B) π (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

Ans. (C)

Sol.



The rectangle will be symmetric about $x = \frac{\pi}{4}$

A (x, 2 sin 2x)

$AB = \frac{\pi}{2} - 2x = CD$

$AD = 2\sin 2x$

$P = 4\sin 2x + \pi - 4x$

$\frac{dP}{dx} = 8 \cos 2x - 4 = 0$

$\cos 2x = \frac{1}{2} \Rightarrow 2\pi = \frac{\pi}{3} \Rightarrow x = \frac{\pi}{6}$

Area = $(2 \sin 2x)\left(\frac{2\pi}{2} - 2x\right)$

Put $x = \frac{\pi}{6}$

$\text{Area}\Big)_{x=\frac{\pi}{6}} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

$= \frac{\sqrt{3}\pi}{6} = \frac{\pi}{2\sqrt{3}}$

SECTION-2 (Maximum Marks : 24)

* This section contains **SIX** (06) questions.
* Each question has **FOUR** options. ONE OR MORE THAN ONE of these four option(s) is (are) correct answer(s).
* For each question, choose the option(s) corresponding to (all) the correct answer(s).
* Answer to each question will be evaluated according to the following marking scheme.

Full Marks : +4 If only (all) the correct option(s) is (are) chosen.

Partial Marks : +3 If all the four options are correct but ONLY three options are chosen.

Partial Marks : +2 If three or more options are correct but ONLY two options are chosen and both of which are correct.

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option.

Zero Marks : 0 If none of the options is chosen (i.e. the question is unanswered).

Negative Marks : –2 In all other cases.

7. Let the function $f : R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g : R \to R$ be an arbitrary function. Let $fg : R \to R$ be the product function defined by $(fg)(x) = f(x)\,g(x)$. Then which of the following statements is/are TRUE?

(A) If g is continuous at x =1, then fg is differentiable at x =1

(B) If fg is differentiable at x = 1, then g is continuous at x =1

(C) If g is differentiable at x =1, then fg is differentiable at x =1

(D) If fg is differentiable at x =1, then g is differentiable at x =1

Ans. (A,C)

Sol. $f(x) = x^3 - x^2 + (x - 1)\sin(x)$

$f(1) = 0$

f is continous & differentiable

$f'(1) = 1 + \sin 1$

Let $h(x) = f(x)\,g(x)$, $h(1) = 0$

Differentiability at x = 1

$h'(1^+) = \lim_{h\to\infty} \frac{f(1+h)g(1+h)-0}{h}$

$= f'(1)\,g(1^+)$

$h'(1^-) = \lim_{h\to\infty} \frac{f(1-h)g(1-h)-0}{-h}$

$= f'(1)\,g(1^-)$

(A) If g is continuous at x = 1

$\Rightarrow g(1^+) = g(1) = g(1^-)$

$\Rightarrow h'(1^+) = h'(1^-)$

(B) If $h'(1^+) = h'(1^-)$

$\Rightarrow g(1^+) = g(1^-)$

but g(1) can be different

(C) If g is differentiable at x = 1

$\Rightarrow$ g is continuous at x = 1

from A we can say that C is true

(B) is false so (D) is also false

8. Let M b a 3 × 3 invertible matrix with real entries and let I denote the 3 × 3 identity matrix.

If M^{-1} = adj (adj M), then which of the following statements is / are ALWAYS TRUE ?

(A) M = I (B) det M = 1 (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

Ans. (B,C,D)

Sol. $M^{-1} = \text{adj}(\text{adj}(M))$

$M^{-1} = |M|\ M$

$I = |M|\ M^2$ ....(1)

$|I| = |\,|M|\ M^2|$

$I = |M|^3\ |M|^2$

$|M|^5 = 1 \Rightarrow |M| = 1$ ....(2)

$\Rightarrow$ Put in (1)

$M^2 = I \Rightarrow M^{-1} = M$

$\frac{\text{adj}(M)}{|M|} = M$

adj (M) = M

$\Rightarrow (\text{adj}(M))^2 = I$

9. Let S be the set of all complex numbers z satisfying $\left|z^2 + z + 1\right| = 1$. Then which of the following statements is/are TRUE?

(A) $\left|z + \frac{1}{2}\right| \le \frac{1}{2}$ for all $z \in S$

(B) $\left|z\right| \le 2$ for all $z \in S$

(C) $\left|z+\frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$

(D) Then set S has exactly four elements

Ans. (B,C)

Sol. $\left|z^2+z+1\right|=1$

$z^2 + z + 1 = e^{i\theta}$

$$z=\frac{-1\pm\sqrt{4e^{i\theta}-3}}{2}$$

$$z+\frac{1}{2}=\pm\frac{1}{2}\sqrt{(4\cos\theta-3)+(4\sin\theta)i}$$

$$\left|z+\frac{1}{2}\right|=\frac{1}{2}\left((4\cos\theta-3)^2+(4\sin\theta)^2\right)^{\frac{1}{4}}$$

$$\left|z+\frac{1}{2}\right|=\frac{1}{2}\left(25-24\cos\theta\right)^{\frac{1}{4}}$$

$$\left|z+\frac{1}{2}\right|\in\left[\frac{1}{2},\frac{\sqrt{7}}{2}\right]$$

$$\left|z+\frac{1}{2}\right|\geq\frac{1}{2}$$

$$2z=-1\pm\sqrt{(4\cos\theta-3)+(4\sin\theta)i}$$

$|2z|\leq 1+\sqrt{7}$ (triangle inequality)

$$|z|\leq\frac{1+\sqrt{7}}{2}$$

$|z| \leq 2$

10. Let x, y and z be positive real numbers. Suppose x, y and z are the length of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\frac{X}{2}+\tan\frac{Z}{2}=\frac{2y}{x+y+z}$$

then which of the following statements is/are TRUE?

(A) $2Y = X + Z$

(B) $Y = X + Z$

(C) $\tan\frac{X}{2} = \frac{x}{y+z}$

(D) $x^2 + z^2 - y^2 = xz$

Ans. (B,C)

Sol. $\tan\left(\frac{X}{2}\right) + \tan\left(\frac{Z}{2}\right) = \frac{2y}{x+y+z}$

$$\frac{\Delta}{s(s-x)} + \frac{\Delta}{s(s-z)} = \frac{2y}{2s}$$

$$\Delta\left(\frac{1}{s-x} + \frac{1}{s-z}\right) = y$$

$$\frac{\Delta\left((s-z)+(s-x)\right)}{(s-x)\ (s-z)} = y$$

$$\frac{\Delta y}{(s-x)(s-z)} = y$$

$$\cot\left(\frac{Y}{2}\right) = 1$$

$$\frac{Y}{2} = \frac{\pi}{4} \Rightarrow Y = \frac{\pi}{2}$$

$\Rightarrow \quad Y = X + Z = \frac{\pi}{2} \qquad \Rightarrow$ A is false / B is true

(c) $\tan\left(\frac{X}{2}\right) = \frac{x}{y+z}$

$$\frac{x}{\sin(X)} = \frac{y}{\sin(90^\circ)} = \frac{z}{\sin(90^\circ - x)}$$

$$\text{RHS} = \frac{y\sin(X)}{y + y\cos X} = \frac{\sin(X)}{1+\cos(X)}$$

$= \tan\left(\frac{X}{2}\right) = \text{LHS}$

(b) $x^2 + z^2 - y^2 = xz$

$\frac{x^2 + z^2 - y^2}{2zx} = \frac{1}{2}$

$\cos Y = \frac{1}{2} \Rightarrow Y = \frac{\pi}{3}$ D is false

11. Let L_1 and L_2 be the following straight lines.

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3} \text{ and } L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}.$$

Suppose the straight line

$$L : \frac{x-\alpha}{l} = \frac{y-1}{m} = \frac{z-\gamma}{-2}$$

lies in the plane containing L_1 and L_2, and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

(A) $\alpha - \gamma = 3$

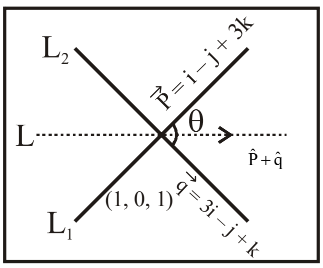
(B) $l + m = 2$

(C) $\alpha - \gamma = 1$

(D) $l + m = 0$

Ans. (A,B)

Sol.



$\vec{p} \cdot \vec{q} = 1 > 0 \Rightarrow \theta \in \left(0, \frac{\pi}{2}\right)$

$\hat{p}+\hat{q}=\frac{1}{\sqrt{11}}(-2i-2j+4k)$

DR's of L = –2, –2, 4

OR

= 1, 1, –2

$L:\frac{x-\alpha}{\ell}=\frac{y-1}{m}=\frac{z-\gamma}{-2}$

$\ell = 1$

m = 1

Passes through (1, 0, 1)

$\frac{1-\alpha}{1}=\frac{0-1}{1}=\frac{1-\gamma}{-2}$

$\alpha = 2$

$\gamma = -1$

$\alpha - \gamma = 3$ (A)

$l + m = 2$ (B)

12. Which of the following inequalities is/are TRUE?

(A) $\int_0^1 x\cos x\ dx \ge \frac{3}{8}$

(B) $\int_0^1 x\sin x\ dx \ge \frac{3}{10}$

(C) $\int_0^1 x^2\cos x\ dx \ge \frac{1}{2}$

(D) $\int_0^1 x^2\sin x\ dx \ge \frac{2}{9}$

Ans. (A,B,D)

Sol. $\text{Cos}\,x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+....$

$\Rightarrow\ \cos x \ge 1-\frac{x^2}{2}$

(A) $\int_0^1 x\cos\ dx \ge \int_0^1 x\left(1-\frac{x^2}{2}\right)dx$

$$\geq \left(\frac{x^2}{2} - \frac{x^4}{8}\right)_0^1$$

$$\geq \frac{3}{8}$$

(B) $\sin x \geq x - \frac{x^3}{3!}$

$$\int_0^1 x \sin x \, dx \geq \int_0^1 \left(x^2 - \frac{x^4}{6}\right) dx$$

$$\geq \left(\frac{x^3}{3} - \frac{x^5}{30}\right)_0^1$$

$$\geq \frac{3}{10}$$

(C) $\cos x \leq 1$

$$\int_0^1 x^2 \cos x \, dx \leq \int_0^1 x^2 \, dx$$

$$\leq \left(\frac{x^3}{3}\right)_0^1$$

$$\leq \frac{1}{3}$$

(D) $\sin x \geq x - \frac{x^3}{3!}$

$$\int_0^1 x^2 \sin x \, dx \geq \int_0^1 \left(x^3 - \frac{x^5}{6}\right) dx$$

$$\geq \left(\frac{x^4}{4} - \frac{x^6}{36}\right)_0^1$$

$$\geq \frac{2}{9}$$

SECTION-3 (Maximum Marks : 24)

* This section contains SIX (06) questions. The answer to each question is a NUMERICAL VALUE.
* For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places truncate/round-off the value to TWO decimal placed.
* Answer to each question will be evaluated according to the following marking scheme :

  Full Marks : +4 If ONLY the correct numerical value is entered.

  Zero Marks : 0 In all other cases.

13. Let m be the minimum possible value of $\log_3(3^{y_1}+3^{y_2}+3^{y_3})$, where y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where x_1, x_2, x_3 are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is _____

Ans. 8

Sol. $\frac{3^{y_1}+3^{y_2}+3^{y_3}}{3} \geq \left(3^{y_1}3^{y_2}3^{y_3}\right)^{\frac{1}{3}}$

$3^{y_1}+3^{y_2}+3^{y_3} \geq 3\left(3^{\frac{y_1+y_2+y_3}{3}}\right)$

$3^{y_1}+3^{y_2}+3^{y_3} \geq 3^4$

$\log_3(3^{y_1}+3^{y_2}+3^{y_3}) \geq 4$

$\Rightarrow m = 4$

$\frac{x_1+x_2+x_3}{3} \geq \left(x_1x_2x_3\right)^{\frac{1}{3}}$

$3 \geq \left(x_1x_2x_3\right)^{\frac{1}{3}}$

$27 \geq x_1x_2x_3$

$\log_3(27) \geq \log_3(x_1x_2x_3)$

$\Rightarrow M = 3$

$\log_2(m^3) + \log_3(M^2) = \log_2(64) + \log_3(9)$

$= 6 + 2 = 8$

14. Let $a_1, a_2, a_3,...$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3,...$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$2(a_1 + a_2 + ... + a_n) = b_1 + b_2 + ... + b_n$

holds for some positive integer n, is _____

Ans. 1

Sol. 2 (sum of first n–terms of AP) = (sum of first n–terms of GP)

$$\Rightarrow 2\left(\frac{n}{2}[2c+(n-1)2]\right) = c\cdot\left(\frac{2^n-1}{2-1}\right)$$

$\Rightarrow 2nc + 2n^2 - 2n = c2^n - c$

$c = \dfrac{2n^2 - 2n}{2^n - 2n - 1}$ = Positive Integer (since a_i & b_i are sequences of positive integers)

Now for n = 1 ; c = 0
n = 2 ; c = – 4
n = 3 ; c = 12

$n = 4; c = \dfrac{24}{7}$

$n = 5; c = \dfrac{40}{21}$

$n = 6; c = \dfrac{60}{51}$

And for all $n \geq 7$; $c < 1$
So no integral value of c can be obtained further.

So only possible value of c = 12.

15. Let $f:[0,2] \rightarrow R$ be the function defined by

$$f(x) = (3 - \sin(2\pi x))\sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right).$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is _____

Ans. 1

Sol. $f(x) = (3 - \sin 2\pi x)\ \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right)$

Let $\pi x - \frac{\pi}{4} = \pi y \Rightarrow x = y + \frac{1}{4}$

$$f(x) = \left(3 - \sin 2\pi\left(y + \frac{1}{4}\right)\right)\ \sin y\,\pi - \sin\left(3\pi\left(y + \frac{1}{4}\right) + \frac{\pi}{4}\right)$$

$$= \left(3 - \sin\left(2\pi y + \frac{\pi}{2}\right)\right)\sin y\pi - \sin\left(3\pi y + \frac{3\pi}{4} + \frac{\pi}{4}\right)$$

$= (3 - \cos 2\pi y)\ \sin\pi\, y + \sin(3\pi y)$

$= (3 - (1 - 2\sin^2 \pi y))\sin \pi y + 3\sin \pi y - 4\sin^3 \pi y$

$= 5\sin \pi y + 2\sin^3 \pi y = \sin \pi y\,(5 + 2\sin^2 \pi y)$

Since $f(x) \geq 0$ $\sin \pi y \geq 0$

$$\Rightarrow \sin \pi\left(x - \frac{1}{4}\right) \geq 0 \Rightarrow \pi\ x - \frac{\pi}{4} \in [2n\pi, (2n + 1)\pi]$$

$$x \in [0, 2] \quad \Rightarrow \pi x - \frac{\pi}{4} \in \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$$

From both of these,

$$\pi x - \frac{\pi}{4} \in [0, \pi]$$

$$\Rightarrow x \in \left[\frac{1}{4}, \frac{5}{4}\right]$$

So, $\alpha = \frac{1}{4}, \beta = \frac{5}{4}$

$$\beta - \alpha = \frac{5}{4} - \frac{1}{4} = 1$$

16. In a triangle PQR, let $\vec{a} = \overrightarrow{QR}, \vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If

$$|\vec{a}| = 3, \quad |\vec{b}| = 4 \text{ and } \frac{\vec{a}\cdot(\vec{c} - \vec{b})}{\vec{c}\cdot(\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|},$$

then the value of $|\vec{a}\times\vec{b}|^2$ is ________

Ans. 108

Sol. $\vec{a}=\overrightarrow{QR},\ \vec{b}=\overrightarrow{RP},\ \vec{c}=\overrightarrow{PQ}$

$\Rightarrow \vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\Rightarrow \vec{c}=-\vec{a}-\vec{b}$

Replacing in the given equation,

$$\frac{\vec{a}\cdot\left(-\vec{a}-\vec{b}-\vec{b}\right)}{\left(-\vec{a}-\vec{b}\right)\cdot\left(\vec{a}-\vec{b}\right)}=\frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}=\frac{3}{3+4}$$

$$\Rightarrow \frac{|\vec{a}|^2+2\vec{a}\cdot\vec{b}}{|\vec{a}|^2-|\vec{b}|^2}=\frac{3}{7}$$

$$\Rightarrow \frac{9+2\,\vec{a}\cdot\vec{b}}{9-16}=\frac{3}{7}$$

Now from Lagrange's Identity

$$|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2-\left(\vec{a}\cdot\vec{b}\right)^2$$

$$=9\times16-(-6)^2$$

$$=144-36=108$$

17. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by

$$S = \{(x^2-1)^2(a_0+a_1x+a_2x^2+a_3x^3):\ a_0, a_1, a_2, a_3\in R\}.$$

For a polynomial f, let f' and f'' denote its first and second order derivatives, respectively. Then the minimum possible value of $\left(m_{f'}+m_{f''}\right)$, where $f\in S$, is _____

Ans. 5

Sol. $f(x) = (x^2-1)^2(a_3x^3+a_2x^2+a_1x+a_0)$

$f'(x) = 2(x^2-1)\,2x\,(a_3x^3+a_2x^2+a_1x+a_0)+(x^2-1)^2(3a_3x^2+2a_2x+a_1)$

$= (x^2 - 1)\,[4x\,(a_3x^3 + a_2x^2 + a_1x + a_0) + (x^2 - 1)\,(3a_3\,x^2 + 2a_2\,x + a_1)]$

$f'(1) = 0 = f'(-1)$

Also, From Rolle’s Theorem on f(x) in [–1, 1],

$f(-1) = 0 = f(1)$

and f(x) is a polynomial function, so it is continuous and differentiable

$\Rightarrow$ There exists at least one c between –1 and 1 such that $f'(c) = 0$

So $f'(x)$ has at least 3 roots : –1, c, 1

So $m_{f'} = 3$

From Rolle’s Theorem, between two roots of f’, at least one root of f’’ will lie.

$\Rightarrow$ minimum no. of roots of $f'' = 2$.

So $m_{f''} = 2$

Min $(m_{f'} + m_{f''}) = 3 + 2 = 5$

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit $\lim_{x\to 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$ is equal to a nonzero real number, is _____

Ans. 1

Sol. $\lim_{x\to 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a} \neq 0$

$\lim_{x\to 0^+} \frac{e^{\frac{1}{x}ln(1-x)} - e^{-1}}{x^a} \neq 0$

$\lim_{x\to 0^+} \frac{e^{-1}(e^{1+\frac{1}{x}ln(1-x)} - 1)}{x^a} \neq 0$

$\lim_{x\to 0^+} \frac{1+\frac{1}{x}ln(1-x)}{ex^a} \neq 0$

$\lim_{x\to 0^+} \frac{x+ln(1-x)}{ex^{a+1}} \neq 0$

$\lim_{x\to 0^+} \frac{x+\left(-x-\frac{x^2}{2}....\right)}{ex^{a+1}} \neq 0$

$\Rightarrow a+1=2$

$\Rightarrow a=1$

if $a<1 \Rightarrow \lim_{x\to 0^+} \frac{x+\left(-x-\frac{x^2}{2}....\right)}{ex^{a+1}} = 0$

if $a>1$ $\lim_{x\to 0^+} \frac{x+\left(-x-\frac{x^2}{2}....\right)}{ex^{a+1}} = \text{DNE}$